Quantum Electrodynamics

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Mir Publishers Moscow

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ARSENII SOKOLOV (1910–1986) was an outstanding Soviet theoretical physicist. He was responsible for fundamental studies in quantum field theory, the physics of elementary particles, the theory of gravitation, the theory of electron accelerators and the theory of radiation emitted by high-energy particles moving in strong electromagnetic fields. Prof. Sokolov developed original mathematical methods for using generalized functions in field theory and effective approximation methods for solving a broad range of the equations of mathematical physics. He became especially known for his studies in the theory of synchrotron radiation, a unique physical phenomenon widely used in science and technology.

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Preface

This book is an expanded version of lectures that we have delivered over the years at the Physics Faculty of Moscow University. The aim of the lectures is to give students specializing in theoretical physics the basics of the theory of classical and quantum fields and an understanding of how to calculate electrodynamical processes. To a great extent we have preserved the structure of the two-semester lecture course. The first two chapters discuss classical field theory and electrodynamics, while the third and fourth are devoted to quantum electrodynamics. We start by formulating the problem of describing classical systems with an infinite number of degrees of freedom via the methods of Lagrange and Hamilton. Along with expounding the theory of fields we felt it necessary to devote some time and space to selected questions of the theory of groups, without a knowledge of which it is impossible to understand not only the Lorentz-invariant structure of field equations but also a modern description of the interactions of elementary particles on the basis of the theory of gauge symmetries.

The discussion of questions of invariance and laws of field transformations is carried out parallel with consideration of the methods of field theory. After proving Noether's theorem, which unites the methods of the theory of groups with the canonical methods of field theory, we give examples of the main fields that represent the Lorentz group, the scalar field, the massive vector field, and the Dirac field. The electromagnetic field is introduced as a gauge, or compensating, field, which ensures the gauge symmetry of the charged fields. The electromagnetic field is generalized to the Yang-Mills fields, which ensure the invariance of material fields under non-Abelian gauge transformations. The last section of Chapter I is devoted to these questions.

Since the student gets acquainted with classical electrodynamics in the general course in theoretical physics, in Chapter II we discuss only some special topics of classical electrodynamics, focusing attention on solution of the wave equations for the components of electromagnetic potentials and development of the classical theory of radiation.

On the basis of the general theory we consider the example of synchrotron radiation, an important physical phenomenon that has found wide application in science and technology.

Chapters III and IV provide the basic tools of quantum electrodynamics. First we quantize the free electromagnetic field in Coulomb gauge and Lorentz gauge (the latter is relativistically invariant). We give an idea of canonical quantization of fields and of the commutation and causal Green functions of the electromagnetic field.

After quantization of the free Dirac field, we formulate the problem of describing interacting electromagnetic and electron-positron fields. Next we build an invariant perturbation theory. Finally, at the end of Chapter III we formulate Feynman's rules that enable constructing the amplitudes of processes of quantum electrodynamics.

The last chapter (Chapter IV) is devoted to calculations of the main processes of interactions of electrons and photons in lower-order perturbation theory. Great attention is paid here to developing the quantum theory of radiation. We detail the synchrotron radiation from the viewpoint of quantum theory.

This phenomenon reveals such a wealth of physical aspects that we have thought it appropriate to also include the results of the main experiments involved in its observation. We also note the main fields of application of synchrotron radiation that follow from its unique properties.

A special section in this chapter singles out briefly the principal difficulties of quantum electrodynamics, that is, the divergence of the terms in perturbation-theory series and the renormalization of the electromagnetic coupling constant and electron mass, and explains how to cope with these difficulties.

The present course is intended for university students of physics, but we believe that it can be used by students of other institutions of higher learning with courses in theoretical physics.

A. A. Sokolov
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Chapter I

Classical Field Theory

1. The Variational Principle in Field Theory

The most effective methods for describing the motion of a system with a finite number of degrees of freedom are those of Lagrange and Hamilton, as the reader should know from the course in theoretical mechanics. The advantage of these methods is their universality, which makes possible a unified description (through appropriate dynamical equations) of any system that can be described via a Lagrange or Hamilton function.

The most general dynamical description of a system with a finite number of degrees of freedom can be obtained by placing the variational principle of Hamilton at the base of such a description. In this case the Lagrange equations are derived as the Euler variational equations which ensure that the action functional will have an extremum.

Fields are physical objects with an infinite number of degrees of freedom, which means they are distributed, or continuous, systems (continuum). To describe such systems, we can also employ the methods of Lagrange and Hamilton based on the variational principle; however, the physical meaning of the Lagrange and canonical variables used in continuum theory is not as obvious as in theoretical mechanics.

1.1 The Linear Chain

Before we turn to the problem of constructing field theory, it is instructive to consider the limiting transition from the mechanics of a system of material particles to continuum mechanics. Suppose we have a linear closed chain (a loop) consisting of N identical particles of mass m each connected by a system of springs (Figure 1.1). Suppose that these springs, which represent the elastic interaction between the particles constituting the chain, have the same spring constant k. We will also assume that in equilibrium the chain is a ring of a radius R and the distances between the neighboring particles

are the same and equal to a. Deviations from equilibrium are allowed only along the circumference of the ring and so are one-dimensional. If we denote the deviation of the sth particle by u_s, we can write the kinetic energy of the chain as

$$T = \sum_{s=1}^{N} \frac{m}{2} \dot{u}_s^2 \tag{1.1}$$

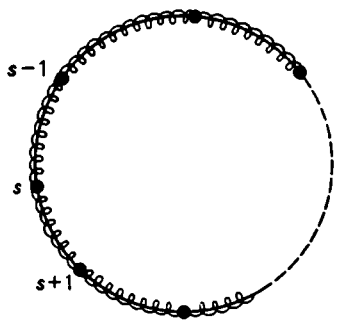

Figure 1.1

and the potential energy as

$$V = \sum_{s=1}^{N} \frac{k}{2} (u_{s+1} - u_s)^2, \tag{1.2}$$

where we have put $\dot{u}_s = du_s/dt$ and introduced the obvious periodicity condition $u_{N+1} = u_1$.

The Lagrange function, or Lagrangian, of the system can then be written thus:

$$L = T - V = \sum_{s=1}^{N} \frac{m}{2} \dot{u}_s^2 - \sum_{s=1}^{N} \frac{k}{2} (u_{s+1} - u_s)^2, \tag{1.3}$$

and the integral

$$S = \int_{t_1}^{t_2} L \, dt \tag{1.4}$$

is called the action.

The equations of motion for the chain are obtained from the condition that

$$\delta S = \delta \int_{t_1}^{t_2} L \, dt = 0 \tag{1.5}$$

at $\delta u_s(t_1) = \delta u_s(t_2) = 0$, $s = 1, 2, \ldots, N$, and are called the Lagrange-Euler equations:

$$\frac{\partial L}{\partial u_s} - \frac{d}{dt} \frac{\partial L}{\partial \dot{u}_s} = 0. \tag{1.6}$$

The set of solutions of these equations, $u_s = u_s(t)$, determines the so-called "straight path" along which the action is stationary and the system performs its true motion.

In the case at hand, according to (1.3), we can write (1.6) explicitly:

$$m\ddot{u}_s = k\,(u_{s+1} - u_s) - k\,(u_s - u_{s-1}), \quad s = 1,\ 2,\ \ldots,\ N. \tag{1.7}$$

To go over to the case where the mass of the chain is distributed continuously over the ring, we consider the limit where $a \to 0$ and $m \to 0$, with the linear density $\rho = m/a$ and Young's modulus $\varepsilon = ka$ remaining finite. Since the size of the chain does not change in the passage to the limit, the number of particles N and hence the number of the degrees of freedom tend to infinity. The "s" that labels the particle is replaced with a continuous variable x, which specifies the position of the particle on the ring, that is,

$$s \to x, \quad a \to dx, \tag{1.8}$$

and the displacement $u_s(t)$ becomes a function of two continuous variables, x and t, and satisfies the periodicity condition:

$$u_s(t) \to u(x,\ t), \quad u(x + 2\pi R,\ t) = u(x,\ t). \tag{1.9}$$

Then

$$u_s - u_{s-1} \to a\frac{\partial u}{\partial x}, \quad (u_{s+1} - u_s) - (u_s - u_{s-1}) \to a^2\frac{\partial^2 u}{\partial x^2}, \tag{1.10}$$

and the system of equations (1.7) becomes a partial differential equation for $u = u(x,\ t)$:

$$\rho\frac{\partial^2 u}{\partial t^2} - \varepsilon\frac{\partial^2 u}{\partial x^2} = 0. \tag{1.11}$$

Mathematically the variables t and x are of equal status. This manifests itself in the fact that the time derivative du_s/dt in the limit $N \to \infty$ must be written as a partial derivative: $du_s/dt \to \partial u/\partial t$. The resulting wave equation (1.11) describes elastic longitudinal waves that propagate along the ring with a speed of $(\varepsilon/\rho)^{1/2}$.

Let us now consider the Lagrangian of the chain

$$L = \sum_{s=1}^{N}\left(\frac{1}{2}\,a\,\frac{m}{a}\,\dot{u}_s^2 - a\,\frac{k}{2}\,\frac{(u_{s+1} - u_s)^2}{a^2}\,ak\right). \tag{1.12}$$

In the limit $N \to \infty$ the sum over s is replaced with an integral,

$$\sum_s = \frac{1}{a}\int dx,$$

and instead of (1.12) we get

$$L = \int dx\left[\frac{1}{2}\,\rho\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\,\varepsilon\left(\frac{\partial u}{\partial x}\right)^2\right] = \int dx\,\mathcal{L}, \tag{1.13}$$

where $\mathscr{L}$ is the Lagrangian density. The action in the case at hand is expressed through a double integral:

$$S = \int dt \int dx \mathscr{L}. \tag{1.14}$$

If we now (formally) write the Lagrange-Euler equation for $\mathscr{L}$, considering x and t as having equal status and being the independent variables,

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \mathscr{L}}{\partial \frac{\partial u}{\partial t}} - \frac{\partial}{\partial x}\frac{\partial \mathscr{L}}{\partial \frac{\partial u}{\partial x}} = 0, \tag{1.15}$$

and take the explicit form for $\mathscr{L}$ from (1.13), we again arrive at the wave equation (1.11). This forces us to conclude that the variational principle can be put at the base of continuum theory, as it was in the theory of systems with a finite number of degrees of freedom.

The example we have chosen is characteristic in that the sums and integrals that enter the definitions of V, T, and L are taken over the entire length of the ring and therefore are finite. Such a situation, as a rule, occurs when one has to solve problems dealing with the motion of continuous media within a finite volume. The periodicity condition, which is quite natural in our example, may be replaced with much more complicated conditions that ensure the finiteness of the energy of the system. In field theory, however, a more common case is the one involving infinitely large volumes. The requirement that the volume integrals for the energy be finite is then ensured by the condition that the functions drop off at infinity sufficiently rapidly.* At the same time it is also possible to partition the infinite volume into finite elementary periodicity volumes, at whose boundaries the field values coincide, and to consider the field only in one such elementary volume. In essence such an approach is equivalent to studying the field at the surface of a three-dimensional torus, which is an obvious generalization of the one-dimensional ring-shaped chain we have just considered.

1.2 The General Variational Problem

In accord with the conclusion made in Section 1.1, let us consider the generalization of the variational problem corresponding to the dynamics of a continuum.

Suppose that the action is given in the general form

$$S = \int_V d^n x \mathscr{L}\left(x;\ u,\ \frac{\partial u}{\partial x}\right), \tag{1.16}$$

* In quantum field theory we encounter a specific divergence in the energy, a divergence that is removed by introducing the so-called normal products of operators (see below).

that is, in the form of a functional of $u(x)$, where x stands for the entire set of the n independent variables x^ν ($\nu = 0, 1, 2, \ldots, n-1$) and u for the set of functions $u^A(x)$ ($A = 1, 2, \ldots, f$). Integration is carried out over a region of the n-dimensional space to which the x^ν belong. Note that in (1.16) there is no dependence on $\partial^2 u/\partial x^2$ or higher-order derivatives. This restriction is not of principal importance and our discussion could be generalized so as to include higher-order derivatives in (1.16), but this would lead to equations of an order higher than the second, which is an extremely rare situation.

Let us assume that at the boundary of the domain of integration the functions u^A are selected in a definite manner (the boundary conditions), while inside the domain the variations of u^A can be assumed arbitrary,

$$u^A = u^A(x) \to u^A(x) + \delta u^A(x).$$

Then

$$\delta \int_V d^n x \, \mathscr{L}(x;\ u,\ \partial u/\partial x) = 0, \tag{1.17}$$

with $\delta u|_\Gamma = 0$ at the boundary Γ of the domain of integration, constitutes the necessary and sufficient condition for the real motion of the system, that is, that $u^A = u^A(x)$ represents the true motion under the given boundary conditions. This constitutes the variational principle for distributed systems (the Hamilton principle).

We now wish to derive the differential equations of the true motion. To this end we write out the variation of S in explicit form:

$$\delta S = \int_V d^n x \left(\frac{\partial \mathscr{L}}{\partial u^A} \delta u^A + \frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \delta u^A_{,\nu} \right). \tag{1.18}$$

Here and subsequently we use the common notation for derivatives, $u^A_{,\nu} \equiv \partial u^A/\partial x^\nu$, and employ the usual summation convention for repeated indices,

$$\frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \delta u^A_{,\nu} \equiv \sum_{A=1}^{f} \sum_{\nu=0}^{n-1} \frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \delta u^A_{,\nu}.$$

Since the independent coordinates x are not varied, variation is commutative with the operation of finding the derivative,

$$\delta \frac{\partial u^A}{\partial x^\nu} = \frac{\partial}{\partial x^\nu} \delta u^A.$$

When applied to (1.18) this yields

$$\delta S = \int_V d^n x \left[\frac{\partial \mathscr{L}}{\partial u^A} \delta u^A + \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \delta u^A \right) - \frac{\partial}{\partial x^\nu} \left(\frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \right) \delta u^A \right], \tag{1.19}$$

which we have written in a form that isolates the divergence of a quantity. This immediately enables us to employ the divergence theorem, which states that the volume integral of the divergence of a (vector) quantity can be transformed into the surface integral of this quantity over the boundary Γ of the volume. But since the later is zero (because $\delta u^A|_\Gamma = 0$), we have

$$\int_V d^n x \frac{\partial}{\partial u^\nu} \left(\frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \delta u^A \right) = \int_\Gamma d\sigma^\nu \left(\frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \delta u^A \right) = 0.$$

As a result condition (1.17) assumes the form

$$\delta S = \int_V d^n x \left(\frac{\partial \mathscr{L}}{\partial u^A} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \right) \delta u^A = 0. \tag{1.20}$$

This together with the fact that the δu^A are arbitrary results in the Lagrange equations (the Euler variational equations):

$$\frac{\partial \mathscr{L}}{\partial u^A} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} = 0 \quad (A = 1, 2, \ldots, f), \tag{1.21}$$

which are the dynamical equations of the motion of a continuous system (a field) with generalized coordinates (functions of the field) u^A. The solutions of these equations, $u^A = u^A(x)$, combined with the appropriate boundary conditions, specify the true motion of the continuous system, the motion that ensures that the action (1.16) is extremal.

In the particular case where $n = 1$ ($\nu = 0$ and $x^0 = t$), Eqs. (1.21) reduce to the equations of theoretical mechanics for discrete systems with f degrees of freedom; for $n = 2$ ($\nu = 0$ and 1, and $x^\nu = t, x$) we arrive at the dynamical equations for a one-dimensional distribution, an example of which ($f = 1$) was considered above (see Eqs. (1.15) and (1.11)).

The left-hand sides of the Lagrange-Euler equations (1.21) are what is known as functional, or variational, derivatives and are denoted thus:

$$\frac{\delta S}{\delta u^A} \equiv \frac{\partial \mathscr{L}}{\partial u^A} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} \tag{1.22}$$

(sometimes $\delta\mathcal{L}/\delta u^A$ is written instead of $\delta S/\delta u^A$). Thus, Eqs. (1.21) imply that the variational derivatives of the action are zero:

$$\delta S/\delta u^A = 0. \tag{1.23}$$

Note that there exists an ambiguity in the definition of $\mathcal{L}$, namely, $\mathcal{L}$ is determined to within the divergence of arbitrary functions $\lambda^\nu = \lambda^\nu(x, u)$, $\nu = 0, 1, 2, \ldots, n-1$. Indeed, the transformation

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \frac{\partial}{\partial x^\nu}\lambda^\nu \tag{1.24}$$

leads to the following transformation in the action:

$$S \to S' = S + \int_\Gamma d\sigma^\nu \lambda_\nu(u),$$

but since

$$\delta \int_\Gamma d\sigma^\nu \lambda_\nu(u) = 0 \quad \text{at} \quad \delta u^A|_\Gamma = 0,$$

we have $\delta S' = \delta S = 0$, and the Lagrange-Euler equations remain valid for $\mathcal{L}'$ too:

$$\frac{\delta S'}{\delta u^A} = \frac{\partial \mathcal{L}'}{\partial u^A} - \frac{\partial}{\partial x^\nu}\frac{\partial \mathcal{L}'}{\partial u^A_{,\nu}} = 0.$$

The variational principle that we have just formulated is the basis for building a theory of motion of dynamical systems. For a field specified in three-dimensional space the number of independent variables x^ν is equal to four. Here both the temporal coordinate $x^0 = t$ and the spatial coordinates x^1, x^2, x^3 participate on an equal basis in the variational principle, which is especially important in constructing a relativistic theory.

1.3 The Hamilton Method

In addition to the Lagrange method, which is based on the Lagrange-Euler equations for the field functions, the generalized coordinates, another canonical method, known as the Hamilton method, is widely used in describing the dynamics of fields and other continuous systems. Let us again begin with the simple model of a one-dimensional elastic medium, the chain in the form of a ring considered above.

Instead of the Lagrange variables u_s, we now introduce for each (sth) particle a canonical pair of variables, the coordinate and the momentum:

$$q_s = u_s, \quad p_s = \frac{\partial L}{\partial \dot{q}_s}. \tag{1.25}$$

We then construct the Hamilton function, or Hamiltonian, of the chain in the following manner:

$$H(p,\ q)=\sum_s p_s\dot{q}_s-L. \tag{1.26}$$

As is known, variables (1.25) obey the following canonical equations:

$$\frac{dq_s}{dt}=\frac{\partial H}{\partial p_s},\quad \frac{dp_s}{dt}=-\frac{\partial H}{\partial q_s},\quad s=1,\ 2,\ \ldots,\ N; \tag{1.27}$$

these equations describe the motion of the point masses along the chain. We now pass to the limit of a continuously distributed mass along the chain:

$$a,\ m\to 0,\ N\to\infty;\ \rho=m/a<\infty,\ \varepsilon=ka<\infty.$$

Then for the canonical momentum we get

$$p_s=\frac{\partial}{\partial\dot{u}_s}\sum_{s'}\frac{1}{2}m\dot{u}_{s'}^2=a\frac{\partial}{\partial\dot{u}_s}\sum_{s'}\frac{1}{2}\rho\dot{u}_{s'}^2\to dx\frac{\partial}{\partial(\partial u/\partial t)}\mathcal{L}, \tag{1.28}$$

where $dx=a$, while the Lagrangian density is defined by formula (1.13):

$$\mathcal{L}=\frac{1}{2}\rho\left(\frac{\partial u}{\partial t}\right)^2-\frac{1}{2}\varepsilon\left(\frac{\partial u}{\partial x}\right)^2. \tag{1.29}$$

Instead of p_s we introduce the density of the canonical momentum through the following formula:

$$\pi=\frac{\partial\mathcal{L}}{\partial\dfrac{\partial u}{\partial t}}. \tag{1.30}$$

Then $p_s=dx\pi$ and for the Hamiltonian we get, in the limit,

$$H=\sum_s p_s\dot{q}_s-L\to\int dx\left(\pi\frac{\partial u}{\partial t}-\mathcal{L}\right)=\int dx\,\mathcal{H}, \tag{1.31}$$

where

$$\mathcal{H}=\pi\frac{\partial u}{\partial t}-\mathcal{L} \tag{1.32}$$

is the Hamiltonian density. Employing the explicit form for $\mathcal{L}$ in (1.29), we can write $\mathcal{H}$ as an explicit function of the canonical variables π, u, and $\partial u/\partial x$:

$$\mathcal{H}=\frac{1}{2}\left(\frac{\pi^2}{\rho}+\varepsilon\left(\frac{\partial u}{\partial x}\right)^2\right). \tag{1.33}$$

Note that in the case at hand there is no explicit dependence of $\mathcal{H}$ on u.

As for the canonical equations (1.27), it is rather difficult to carry out the limiting process for them because the canonical momentum $p_s = a\pi$ tends to zero as $a \to 0$. For this reason we will now derive directly the canonical field equations in a manner similar to the one employed for discrete systems, that is, using Legendre transformations.

Suppose we know the function of the variables x, u, and $\partial u/\partial x$, or the Lagrangian density

$$\mathscr{L} = \mathscr{L}\left(t,\ \mathbf{x};\ u,\ \frac{\partial u}{\partial t},\ \nabla u\right). \tag{1.34}$$

We explicitly isolated the temporal and spatial coordinates and the derivatives with respect to these coordinates in the independent variables of $\mathscr{L}$ on purpose, since in the canonical method they cease to have equal status.

Let us go over from the old variables to new, canonical, variables:

$$t,\ \mathbf{x},\ u^A,\ \nabla u^A,\ \partial u^A/\partial t \to t,\ \mathbf{x},\ u^A,\ \nabla u^A,\ \pi^A,\ \nabla \pi^A, \tag{1.35}$$

where

$$\pi_A = \frac{\partial \mathscr{L}}{\partial u^A_{,t}} \quad (A = 1,\ 2,\ \ldots,\ f). \tag{1.36}$$

To this end we must solve the last equation with respect to $u^A_{,t}$, that is, determine the functional dependence of the old variable $u^A_{,t}$ on the new variables, or

$$\frac{\partial u^A}{\partial t} = f^A(x;\ u,\ \nabla u,\ \pi,\ \nabla \pi). \tag{1.37}$$

Such an equation can be found if Hesse's determinant (the Hessian does not vanish,

$$\det\left(\frac{\partial^2 \mathscr{L}}{\partial u^A_{,t}\, \partial u^B_{,t}}\right) \neq 0,$$

for all x's in the domain of $u^A(x)$.

Let us now construct the generating function of the transformation, the Hamiltonian density

$$\mathscr{H} = \pi_A \frac{\partial u^A}{\partial t} - \mathscr{L} = \mathscr{H}(x;\ u,\ \nabla u,\ \pi,\ \nabla \pi). \tag{1.38}$$

The integral of the Hamiltonian density over the volume,

$$H = \int d^3x\, \mathscr{H} = \int d^3x \left(\pi_A \frac{\partial u^A}{\partial t} - \mathscr{L}\right), \tag{1.39}$$

is the system's Hamiltonian given as a functional of u^A and π_A. Since the integral in (1.39) is over a three-dimensional volume, the

functional derivatives of this integral, in accord with (1.22), can be written thus:

$$\frac{\delta H}{\delta \pi_A} = \frac{\partial \mathscr{H}}{\partial \pi_A} - \nabla \frac{\partial \mathscr{H}}{\partial \nabla \pi_A}, \quad \frac{\delta H}{\delta u^A} = \frac{\partial \mathscr{H}}{\partial u^A} - \nabla \frac{\partial \mathscr{H}}{\partial \nabla u^A}. \tag{1.40}$$

Here, for example,

$$\nabla \frac{\partial \mathscr{H}}{\partial \nabla u^A} \equiv \frac{\partial}{\partial x^k} \frac{\partial \mathscr{H}}{\partial u^A_{,k}},$$

and summation is carried out over $k = 1, 2, 3$.

Using the explicit form for $\mathscr{H}$ given by (1.38), we can first calculate the derivative with respect to π_A:

$$\frac{\partial \mathscr{H}}{\partial \pi_A} = \frac{\partial u^A}{\partial t} + \pi_B \frac{\partial u^B_{,t}}{\partial \pi_A} - \frac{\partial \mathscr{L}}{\partial u^B_{,t}} \frac{\partial u^B_{,t}}{\partial \pi_A}. \tag{1.41}$$

Here, of course, $u^A_{,t}$ is understood to be a function of the canonical variables (see (1.37)). Recalling the definition (1.36) of π_B, we find, instead of (1.41), that

$$\frac{\partial \mathscr{H}}{\partial \pi_A} = \frac{\partial u^A}{\partial t}. \tag{1.42}$$

The second term in the expression for $\delta H/\delta \pi_A$ vanishes since

$$\frac{\partial \mathscr{H}}{\partial \nabla \pi_A} = \pi_B \frac{\partial u^B_{,t}}{\partial \nabla \pi_A} - \frac{\partial \mathscr{L}}{\partial u^B_{,t}} \frac{\partial u^B_{,t}}{\partial \nabla \pi_A} = \pi_B \frac{\partial u^B_{,t}}{\partial \nabla \pi_A} - \pi_B \frac{\partial u^B_{,t}}{\partial \nabla \pi_A} = 0.$$

Thus,

$$\frac{\delta H}{\delta \pi_A} = \frac{\partial u^A}{\partial t}, \tag{1.43}$$

which constitutes Hamilton's first canonical equation for the field.

Next we vary H with respect to u. Let us take the derivative of $\mathscr{H}$ with respect to u^A:

$$\frac{\partial \mathscr{H}}{\partial u^A} = \pi_B \frac{\partial u^B_{,t}}{\partial u^A} - \frac{\partial \mathscr{L}}{\partial u^A} - \frac{\partial \mathscr{L}}{\partial u^B_{,t}} \frac{\partial u^B_{,t}}{\partial u^A} = - \frac{\partial \mathscr{L}}{\partial u^A}.$$

At the same time,

$$\frac{\partial \mathscr{H}}{\partial \nabla u^A} = \pi^B \frac{\partial u^B_{,t}}{\partial \nabla u^A} - \frac{\partial \mathscr{L}}{\partial \nabla u^A} - \frac{\partial \mathscr{L}}{\partial u^B_{,t}} \frac{\partial u^B_{,t}}{\partial \nabla u^A} = - \frac{\partial \mathscr{L}}{\partial \nabla u^A}.$$

Substituting these expressions into the second equation in (1.40), we find that

$$\frac{\delta H}{\delta u^A} = - \frac{\partial \mathscr{L}}{\partial u^A} + \nabla \frac{\partial \mathscr{L}}{\partial \nabla u^A}. \tag{1.44}$$

The right-hand side of (1.44) is minus one multiplied by the functional derivative of the system's Lagrangian:

$$\frac{\delta L}{\delta u^A} = \frac{\partial \mathscr{L}}{\partial u^A} - \nabla \frac{\partial \mathscr{L}}{\partial \nabla u^A}, \quad L = \int \mathscr{L}\, d^3x.$$

This enables us to write (1.44) in a more compact form:

$$\frac{\delta H}{\delta u^A} = -\frac{\delta L}{\delta u^A}. \tag{1.45}$$

This equation is equivalent to the corresponding equation connecting the partial derivatives of H and L with respect to the generalized coordinates in the theory of systems with a finite number of degrees of freedom, $\partial H/\partial q^s = -\partial L/\partial q^s$. For true trajectories we have, in view of the field equations (1.21) (the Lagrange-Euler equations),

$$\frac{\delta L}{\delta u^A} = \frac{\partial}{\partial t} \frac{\partial \mathscr{L}}{\partial u^A_{,t}} = \frac{\partial \pi_A}{\partial t},$$

whereby (1.45) becomes

$$\frac{\delta H}{\delta u^A} = -\frac{\partial \pi_A}{\partial t}, \tag{1.46}$$

and this constitutes Hamilton's second canonical equation.

Thus, by carrying out the transformation of variables (1.35) and by introducing a new function $\mathscr{H}$ instead of $\mathscr{L}$, a function that depends on the canonical variables, we obtained, in view of the dynamical equations for the Lagrange variables u^A, Eqs. (1.43) and (1.46) for the canonical variables. Obviously, the inverse transformation is also possible. To this end, from Eq. (1.42) we find π_A and from (1.38) we find $\mathscr{L}$. In view of Eqs. (1.46) and (1.45), we then get the Lagrange-Euler equations. And so the dynamical description of a system with an infinite number of degrees of freedom can be carried out by two methods, by the Lagrange method and by the Hamilton method, just as in the case of a system with a finite number of degrees of freedom. In the first method, the system is characterized by the generalized "coordinates", or the field functions u^A, which obey the Lagrange-Euler equations. In the second method, the number of the field variables increases due to the introduction of the densities of the canonical momenta, π_A. These variables, u^A and π_A, obey the Hamilton equations

$$\frac{\delta H}{\delta \pi_A} = \frac{\partial u^A}{\partial t}, \quad \frac{\delta H}{\delta u^A} = -\frac{\partial \pi_A}{\partial t}. \tag{1.47}$$

While the Lagrange-Euler equations are second-order differential equations in all the variables t and x^k, the Hamilton equations are first-order in t, with the result that their number is doubled. We see,

therefore, that the time variable plays a special role in the Hamilton method. This violates the equal status of spatial and temporal coordinates inherent in the Lagrange method. At the same time, as quantum mechanics demonstrates, quantization is carried out using the language of canonical variables [3], and therefore quantization of fields requires using the Hamilton method (see below).

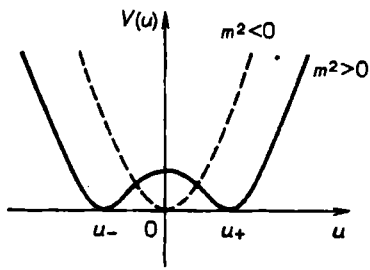

Figure 1.2

Let us again take up the chain example. Using the formula for $\mathscr{L}$, (1.29), we find that

$$\pi = \rho \frac{\partial u}{\partial t}, \quad \mathscr{H} = \frac{1}{2}\left[\frac{\pi^2}{\rho} + \varepsilon \left(\frac{\partial u}{\partial x}\right)^2\right]. \tag{1.48}$$

The Hamilton equations (1.47) then assume the form

$$\frac{\pi}{\rho} = \frac{\partial u}{\partial t}, \quad \varepsilon \frac{\partial^2 u}{\partial x^2} = \frac{\partial \pi}{\partial t}.$$

Substituting the first into the second yields the wave equation (1.11).

Let us take one more example. Suppose that the Lagrangian density of a real field u is given in the form

$$\mathscr{L} = \frac{1}{2}\rho \left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\varepsilon \left(\frac{\partial u}{\partial x}\right)^2 - V(u), \tag{1.49}$$

where $V(u)$ is the potential energy of the field equal to (Figure 1.2)

$$V(u) = \frac{1}{4}\lambda \left(\frac{m^2}{\lambda} - u^2\right)^2 \quad (\lambda > 0). \tag{1.50}$$

We wish to find the density of canonical momentum,

$$\pi = \rho \frac{\partial u}{\partial t}, \tag{1.51}$$

and the density of the field's energy,

$$\mathscr{H} = \pi \frac{\partial u}{\partial t} - \mathscr{L} = \frac{1}{2}\frac{\pi^2}{\rho} + \frac{1}{2}\varepsilon \left(\frac{\partial u}{\partial x}\right)^2 + V(u). \tag{1.52}$$

We request that the density of the field's energy at infinity ($|\mathbf{x}| \to \infty$) be minimal (the classical vacuum), or

$$\mathscr{H} \to \min \mathscr{H} = 0 \quad \text{as} \quad |\mathbf{x}| \to \infty. \tag{1.53}$$

We assume that m^2 is positive (the solid curve in Figure 1.2). The following asymptotic behavior of the field function obeys condition (1.53):

$$|\mathbf{x}| \to \infty, \; u^2 \to \text{constant} = m^2/\lambda, \; \pi \to 0. \tag{1.54}$$

For u we obtain two limits (vacuum solutions): $u \to u_{\pm} = \pm(m^2/\lambda)^{1/2}$. If we take only one solution, say $u_+ = (m^2/\lambda)^{1/2}$, we find that this breaks the symmetry of $\mathscr{H}$ in (1.52) with respect to reflections $u \to -u$, or, as it is usually said, spontaneous symmetry breaking occurs. This is due to the instability near the symmetric solution $u \to 0$, since for u close to zero the potential energy is close to its maximum:

$$u \to 0, \quad V(u) \approx -\frac{m^2}{2}u^2 + \frac{m^4}{4\lambda}.$$

In the limit $m^2 \to 0$, $\lambda \to 0$, $m^2/\lambda < \infty$, the asymmetry of the stable solution is preserved, while for $m^2/\lambda \to 0$ it vanishes.

If m^2 is negative (the dashed curve in Figure 1.2), then $V(u)$ has a minimum at the origin. In this case the solution near the vacuum state $u \to u_0 = 0$ is stable. Therefore, when we go over to negative values of m^2, the symmetry is restored. Such behavior of solutions of nonlinear equations occurs in Landau's theory of second-order phase transitions (the ferromagnetic transition, superconductivity). The quantum theory of such phenomena (the Higgs mechanism) makes it possible to explain the origin of the mass of elementary particles (see Section 11).

2. Lorentz Transformations

We will not discuss gravitational forces, since their consideration is the topic of the general theory of relativity, and this is beyond the scope of our book. We therefore operate within the framework of the special theory of relativity.

As is known, the special theory of relativity is based on Einstein's principle of relativity, which consists of two postulates: (1) all physical laws are invariant with respect to the choice of various inertial frames of reference,* and (2) the speed of propagation of light in a vacuum does not depend on the particular reference frame and is the same in all directions. The numerical value of the speed of light in vacuum, c, is approximately 3×10^{10} cm/s and constitutes a universal physical constant.

This principle, or the two postulates considered simultaneously, leads to the necessity of considering a special type of linear transformations of the four coordinates (three spatial and one temporal) known as the Lorentz transformations.

* Inertial frames of reference are those in which Newton's first law is valid, that is, physical bodies move uniformly and rectilinearly in the absence of external forces.

2.1 Minkowski Space-Time

Let us select a certain inertial reference frame specified by three orthogonal base (unit) vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ (this triple of vectors specifies a Cartesian system of coordinates) and a clock at the origin for measuring the temporal coordinate, or simply time. Then, in this reference frame, the field functions u will depend on four variables, namely, on the three orthogonal coordinates x^k ($k = 1, 2, 3$) of the three-dimensional vector $\mathbf{x} = x^k\mathbf{e}_k$ and the time t.

Using the fact that the speed of light is a universal constant, we introduce a new variable $x^0 = ct$, which enables us to consider the four numbers $x^\mu = x^0, x^1, x^2, x^3$ as the components of a vector x in a four-dimensional linear space:

$$x = x^\mu e_\mu, \tag{2.1}$$

where $e_\mu = e_0, e_1, e_2, e_3$ are the base vectors in this space. Next we introduce a metric in the four-dimensional space (or 4-space), that is, we define the square of the distance between two points in the 4-space connected by vector x in the following manner:

$$x^2 \equiv x \cdot x = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = c^2t^2 - \mathbf{x}^2. \tag{2.2}$$

Using (2.1), we can write the square of vector x as follows:

$$x^2 = x^\mu x^\nu e_\mu e_\nu = x^\mu x^\nu g_{\mu\nu}, \tag{2.3}$$

where, in accordance with (2.2), the metric tensor $g_{\mu\nu}$ is defined as

$$g_{\mu\nu} = g_{\nu\mu} = e_\mu e_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu = 0, \\ -1 & \text{if} \quad \mu = \nu = 1, 2, 3, \\ 0 & \text{if} \quad \mu \neq \nu. \end{cases} \tag{2.4}$$

As the reader can see, the base vectors e_μ are pseudo-orthogonal.

In exactly the same manner, for the scalar product of two vectors x and y we obtain

$$xy = x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3 = x^0y^0 - \mathbf{x} \cdot \mathbf{y} = x^\mu y^\nu g_{\mu\nu}. \tag{2.5}$$

The metric linear space introduced above is called pseudo-Euclidean, since the signs of x^2 and xy can be both plus and minus. In honor of the Russian-German mathematician Hermann Minkowski, who gave a geometric interpretation of the special theory of relativity, this space is called the Minkowski space-time.

Mathematically the postulate of the constancy of the speed of light means that the quadratic form (2.2) is invariant under transitions from one inertial reference frame to another, that is,

$$x^2 = c^2t^2 - \mathbf{x}^2 = \text{inv} = 0. \tag{2.6}$$

The transition from one inertial frame of reference to another is realized by a linear transformation of the spatial and temporal coordinates: $x, y, z, t \to x', y', z', t'$. The linearity ensures that an object in a new reference frame is not accelerated if it is not accelerated in the old system, in accordance with Einstein's principle of relativity.

If the new frame of reference is moving along the x^1 axis with a velocity v in relation to the old reference system, the appropriate coordinate transformation, known as the special Lorentz transformation, is

$$\begin{aligned} x^{1'} &= \gamma (x^1 - \beta x^0), \\ x^{2'} &= x^2, \\ x^{3'} &= x^3, \\ x^{0'} &= \gamma (x^0 - \beta x^1), \text{ where } \beta = v/c, \ \gamma = (1 - \beta^2)^{-1/2}. \end{aligned} \tag{2.7}$$

This type of transformation preserves the invariance of quadratic form (2.2) not only in the particular case (2.6), $x^2 = 0$, which corresponds to light propagation, but in the general case $x^2 \neq 0$. The transformation of coordinates x^1, $x^0 = ct \to x^{1'}$, $x^{0'} = ct'$, (2.7), can also be written as follows:

$$\begin{aligned} x^{1'} &= x^1 \cosh \psi - x^0 \sinh \psi, \\ x^{0'} &= -x^1 \sinh \psi + x^0 \cosh \psi, \end{aligned} \tag{2.8}$$

where $\tanh \psi = \beta$. This is a pseudorotation in the 01-plane through an imaginary angle $i\psi$; such a rotation, obviously, does not change the value of the difference of squares of the coordinates:

$$(x^0)^2 - (x^1)^2 = (x^{0'})^2 - (x^{1'})^2.$$

2.2 The General Lorentz Transformations

The special Lorentz transformation (2.7), which corresponds to a transition from one inertial reference frame to another, is only a special case of a more general Lorentz transformation, that is, general linear transformations that preserve the square of the 4-vector (2.2) in Minkowski's space-time. Let us consider such transformations in greater detail. Suppose the "old" basis e_ν is linearly transformed into a "new" basis e'_μ. Then the new coordinates x'^μ of a vector are connected with the old coordinates x^ν via the linear relation

$$x'^\mu = A^\mu_{\ \nu} x^\nu, \tag{2.9}$$

where $A = \| A^{\mu}_{\nu} \|$ is the matrix of the general Lorentz transformation:

$$A = \begin{pmatrix} A^0_0 & A^0_1 & A^0_2 & A^0_3 \\ A^1_0 & A^1_1 & A^1_2 & A^1_3 \\ A^2_0 & A^2_1 & A^2_2 & A^2_3 \\ A^3_0 & A^3_1 & A^3_2 & A^3_3 \end{pmatrix}.$$

Since vector x can be written both in the old basis and in the new one,

$$x = x^{\nu} e_{\nu} = x'^{\mu} e'_{\mu} = A^{\mu}_{\nu} x^{\nu} e'_{\mu}, \tag{2.10}$$

we can write

$$e_{\nu} = A^{\mu}_{\nu} e'_{\mu},$$

which yields the law of transformation of basis:

$$e'_{\mu} = (A^{-1})^{\nu}_{\mu} e_{\nu}. \tag{2.11}$$

Thus, if the components of a vector x are transformed by means of a matrix $A = \| A^{\mu}_{\nu} \|$, the base vectors are transformed via the transpose of the inverse matrix $(A^{-1})^T = \| (A^{-1})^{\nu}_{\mu} \|$.

A covariant vector may be constructed out of a contravariant vector as follows:

$$x_{\mu} = g_{\mu\nu} x^{\nu} = e_{\mu} e_{\nu} x^{\nu} = e_{\mu} \cdot x. \tag{2.12}$$

Indeed, then $x'_{\mu} = e'_{\mu} \cdot x$, and the components x'_{μ} are related to x_{μ} by the same law as e'_{μ} is related to e_{μ}. Taking into account the explicit form (2.4) of the metric tensor $g_{\mu\nu}$, we get

$$x_0 = x^0, \; x_1 = -x^1, \; x_2 = -x^2, \; x_3 = -x^3. \tag{2.13}$$

In addition to the metric tensor $g_{\mu\nu}$ we can introduce the contravariant tensor $g^{\mu\nu}$, which is the inverse of $g_{\mu\nu}$, or $g_{\mu\sigma} g^{\sigma\nu} = \delta^{\nu}_{\mu}$. The components of the two tensors coincide: $g_{\mu\nu} = g^{\mu\nu}$.

Employing the tensor $g^{\mu\nu}$, we can raise the indices of a covariant vector, thus transforming it into a contravariant vector:

$$x^{\mu} = g^{\mu\sigma} x_{\sigma}. \tag{2.14}$$

We will now establish what condition must the matrix A^{μ}_{ν} of the linear transformation satisfy so that the invariance of the square of a vector, x^2, is ensured. To this end we write the condition $x^2 = \text{inv}$ in explicit form:

$$x'^2 = x^2, \tag{2.15}$$

that is,

$$A^{\mu}_{\nu} A^{\lambda}_{\rho} g_{\mu\lambda} x^{\nu} x^{\rho} = x^{\alpha} x^{\beta} g_{\alpha\beta}. \tag{2.16}$$

From this follows the basic result:

$$A^{\mu}_{\nu} g_{\mu\lambda} A^{\lambda}_{\rho} = g_{\nu\rho}, \tag{2.17}$$

which means that $g_{\mu\nu}$ is not transformed when we go over to new coordinates—this is simply another way of expressing the fact that x^2 is an invariant. Multiplying (2.17) into $g^{\nu\sigma}$ yields

$$A^{\mu}_{\nu} g_{\mu\lambda} g^{\nu\sigma} A^{\lambda}_{\rho} = \delta^{\sigma}_{\rho}, \tag{2.18}$$

that is, the matrix $\mathrm{A}'' = \| A^{\mu}_{\nu} g_{\mu\lambda} g^{\nu\sigma} \|$ is the inverse of $\| A^{\mu}_{\nu} \|$. Let us find the explicit form of A''. First we note that the transition from matrix A to matrix $\mathrm{A}' = \| A^{\mu}_{\nu} g_{\mu\lambda} \|$ is reduced to lowering one index by applying $g_{\mu\lambda}$ and, according to (2.13), reversing the sign of all matrix rows except the first. The transition from A' to A'' is reduced to transposing A' and reversing the sign of all matrix columns except the first. The result is a matrix A'' which has the form

$$\mathrm{A}'' = \begin{pmatrix} A^0_0 & -A^1_0 & -A^2_0 & -A^3_0 \\ -A^0_1 & A^1_1 & A^2_1 & A^3_1 \\ -A^0_2 & A^1_2 & A^2_2 & A^3_2 \\ -A^0_3 & A^1_3 & A^2_3 & A^3_3 \end{pmatrix}. \tag{2.19}$$

We arrive at the following property of the matrix of the general Lorentz transformation: transposing the initial matrix and reversing the signs of matrix elements in the upper row and left column transforms A into the inverse matrix: $\mathrm{A}'' = \mathrm{A}^{-1}$.

Such matrices are said to be pseudo-orthogonal (in contrast to orthogonal matrices in which the transition to the inverse matrix means only transposing the initial matrix; such is the matrix of spatial rotations, which we will discuss later).

Formula (2.18) implies that

$$\det \mathrm{A}'' \cdot \det \mathrm{A} = 1, \tag{2.20}$$

but $\det \mathrm{A}'' = \det \mathrm{A}$ (transposition of a matrix and subsequent reversal of sign of a column and a row do not change the value of the determinant), whereby $\det^2 \mathrm{A} = 1$. This equation has two solutions:

$$\det \mathrm{A} = +1 \quad \text{and} \quad \det \mathrm{A} = -1. \tag{2.21}$$

In both cases $\det \mathrm{A} \neq 0$, and this is proof that each transformation A has an inverse whose matrix is A''.

Note that the condition $|\det \mathrm{A}| = 1$ implies the invariance of an element of volume of 4-space:

$$d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x = |\det \mathrm{A}|\, d^4x = d^4x.$$

The special Lorentz transformation needed for transferring to a reference frame that moves along the x^1 axis with velocity v, (2.7), corresponds to a symmetric matrix A with the following matrix elements being nonzero:

$$A_0^0 = A_1^1 = \gamma, \quad A_2^2 = A_3^3 = 1, \quad A_1^0 = A_0^1 = -\beta\gamma.$$

For the matrix A″ we have

$$(A'')_0^0 = (A'')_1^1 = \gamma, \quad (A'')_2^2 = (A'')_3^3 = 1, \quad (A'')_1^0 = (A'')_0^1 = \beta\gamma.$$

It is clear that A″ is the matrix of the inverse transformation since it differs from A in the sign of the velocity, $\beta \to -\beta$, and, besides, direct multiplication of the two matrices yields A″A = I. The determinant of the matrix of this transformation is unity, or det A = +1.

Let us now consider the main relationship (2.17) for $\nu = \rho = 0$:

$$A_0^\mu g_{\mu\lambda} A_0^\lambda = 1,$$

or in explicit form,

$$(A_0^0)^2 - (A_0^1)^2 - (A_0^2)^2 - (A_0^3)^2 = 1. \tag{2.22}$$

From this it follows that

$$(A_0^0)^2 = 1 + (A_0^1)^2 + (A_0^2)^2 + (A_0^3)^2 \geqslant 1,$$

that is,

$$A_0^0 \leqslant -1 \quad \text{or} \quad A_0^0 \geqslant 1. \tag{2.23}$$

Thus, the general Lorentz transformation, denoted here by L, constitutes four sets of transformations, and the choice of a particular transformation depends on the sign of det A and the inequality in (2.23):

	$L_+^\uparrow$	$L_+^\downarrow$	$L_-^\uparrow$	$L_-^\downarrow$
A_0^0	$\geqslant 1$	$\leqslant -1$	$\geqslant 1$	$\leqslant -1$
det A	$+1$	$+1$	-1	-1

Clearly, by continuously varying the elements of any one of the four sets we cannot go over to another set, that is, these sets are not connected by a continuous transformation. Only $L_+^\uparrow$ contains the unit, or identity, transformation A = I, for which, obviously, $A_0^0 = 1$ and det A = 1. By continuously changing the unit transformation A = I we can obtain all the transformations in set $L_+^\uparrow$. These are known as proper Lorentz transformations and constitute four-dimensional pseudorotations. The invariance under such trans-

formations is the reflection of the fact that space-time is isotropic (if we ignore gravitational interaction).

All the other transformations contain discrete transformations (space-time reflection) and cannot be reduced to continuous transformations.

Let us consider the timelike vector

$$x^2 = (x^0)^2 - \mathbf{x}^2 > 0 \tag{2.24}$$

(for a spacelike vector we have $x^2 = (x^0)^2 - \mathbf{x}^2 < 0$). Suppose that the sign of the temporal component is fixed, say, $x^0 > 0$. Then the Lorentz transformations with $A_0^0 \geqslant 1$ do not change the sign of the temporal component, or $x'^0 > 0$. Indeed,

$$x'^0 = A_0^0 x^0 + A_1^0 x^1 + A_2^0 x^2 + A_3^0 x^3. \tag{2.25}$$

If we combine the Cauchy-Buniakowski inequality with (2.22) and (2.24), we get

$$(A_1^0 x^1 + A_2^0 x^2 + A_3^0 x^3)^2 < [(A_0^0)^2 - 1]\ (x^0)^2 < (A_0^0)^2\ (x^0)^2.$$

This implies that if $A_0^0 \geqslant 1$, then x'^0 given by (2.25) retains the sign of x^0, that is, $x'^0 > 0$ in the case at hand. Lorentz transformations with $A_0^0 \geqslant 1$, which do not change the sign of the temporal component of a timelike vector, are known as orthochronous. Lorentz transformations with $A_0^0 \leqslant -1$ do not retain the sign of x^0 and are called nonorthochronous. Thus, the set $L_\uparrow^\uparrow$ consists of proper orthochronous transformations, also known as restricted Lorentz transformations. The special Lorentz transformations, which fix a transformation from one inertial reference frame to another, obviously belong to the set $L_+^\uparrow$, or proper orthochronous transformations.

Here are some examples of improper (discrete) Lorentz transformations.

(1) Reflection of one of the spatial axes (the P_1 operation). For matrix A we have: $A_0^0 = 1$, $\det A = -1$. Therefore, $P_1 \subset L_\downarrow^\uparrow$.

(2) Space inversion (the P operation): $\mathbf{x} \to \mathbf{x}' = -\mathbf{x}$. For this transformation we also have $A_0^0 = 1$, $\det A = -1$, $P \subset L_\downarrow^\uparrow$. Note that the reflection of any two spatial axes is equivalent to a finite and continuous rotation transformation, which is a particular case of $L_+^\uparrow$. Therefore, P_1 can be obtained continuously from P, and vice versa.

(3) Time inversion (the T operation): $t \to t' = -t$. Here $A_0^0 = -1$, $\det A = -1$, that is, $T \subset L_\downarrow^\downarrow$.

(4) Space and time inversion (the PT operation). Now $A_0^0 = -1$, $\det A = +1$, and hence $PT \subset L_+^\downarrow$.

Note that proper Lorentz transformations are used in classical mechanics and in the classical and quantum field theories. The improper, discrete, operations P, T, and PT find use primarily in quan-

tum mechanics, quantum field theory, and the theory of elementary particles. Any improper transformation, $L_-^\uparrow$ or $L_-^\downarrow$ or $L_+^\downarrow$, can be obtained from the P, T, or PT operations by applying the proper continuous transformation $L_+^\uparrow$, as can easily be seen.

3. The Lorentz Group and the Poincaré Group

Lorentz transformations constitute an example of transformations that form a group. In what follows we often consider various groups of transformations of field functions; we therefore propose a brief outline of the theory of continuous, or topological, groups.

3.1 Lie Groups

Mathematics requires a following definition of an abstract group. A group G is a set of elements $A, B, C, \ldots \in G$ for which the composition law (the product) is defined for any two elements: $AB = C$. The product must obey the following conditions: (1) if $A \in G$ and $B \in G$, then $C = AB \in G$; (2) if $A, B, C \in G$, then $(AB)C = A(BC) \equiv ABC$; (3) there is an element $I \in G$ such that $AI = IA = A$ for any $A \in G$; and (4) for each $A \in G$ there is a $B \in G$ such that $AB = BA = I$. The element $B = A^{-1}$ is said to be the inverse (or inverse element) of A.

The general Lorentz transformations constitute a group. Indeed, a subsequent application of two transformations that do not change the quadratic form x^2 also changes nothing in the quadratic form, there is an identity, or unit, element, and each element has an inverse. Thus, the linear transformations that preserve the invariance of the quadratic form $x^2 = (x^0)^2 - \mathbf{x}^2$ constitute a group known as the general Lorentz group, or the homogeneous Lorentz group.

If to this group we add translations, that is, a translation of the origin of the coordinate system by a constant vector a^μ, we arrive at nonhomogeneous linear transformations $x^\mu \to x'^\mu = A^\mu_\nu x^\nu + a^\mu$. These transformations form the nonhomogeneous Lorentz group, or the Poincaré group. Obviously, the length of the vector connecting two points x and y in Minkowski's space-time remains unchanged under all translation transformations: $x \to x + a$, $y \to y + a$, $(x - y)^2 = \text{inv}$. This property reflects the homogeneity of space-time.

Out of the four sets of transformations into which the general Lorentz group splits, only $L_+^\uparrow$ forms a group, since it contains a unit element. This is the proper Lorentz group.

A more simple example is the $O(3)$ group, which is the group of rotations in three-dimensional space. The matrices of such transformations, A_{ik}, which preserve the length of vector $\mathbf{x}$, are orthogonal:

$A_{ik}A_{kj} = \delta_{ij}$. This group constitutes a subgroup of the proper Lorentz group.

When we speak of the O (3) group or the Lorentz group, we have in mind certain objects that undergo the transformations, that is, the coordinates of vectors in three-dimensional Euclidean space and in four-dimensional Minkowski's space-time, as well as matrices of a certain type: three-dimensional orthogonal and four-dimensional pseudo-orthogonal matrices, respectively. All this fixes the concrete (vector) representations of these groups. At the same time, the elements of these groups may have, as will be shown later, other realizations.

In general, if to each element A of group G we assign a linear operator T_A, that is, $A \in G$, $A \to \mathrm{T}_A$, which acts in a linear n-dimensional space R, then it is said that an n-dimensional linear representation of group G is fixed. Space R is called the representation space of group G.

The above-noted correspondence means that (1) if $A \in G$ and $B \in G$, then $\mathrm{T}_A\mathrm{T}_B = \mathrm{T}_{AB}$, (2) the operator of the unit element is the unit operator, $\mathrm{T}_I = \mathrm{I}$, and (3) the operator of the inverse element is the inverse operator, $\mathrm{T}_{A^{-1}} = \mathrm{T}_A^{-1}$.

In a finite-dimensional space, operators can be represented by matrices, and then we talk of a matrix representation. If e_α is a basis in R, that is, $x = e_\alpha x^\alpha$ and T is the operator of the representation, then

$$e'_\alpha = \mathrm{T}e_\alpha = e_\beta D^\beta_\alpha, \quad x'^\beta = D^\beta_\alpha x^\alpha, \tag{3.1}$$

where $\| D^\beta_\alpha \|$ is the matrix of operator T. If we go over to a new basis in R, or

$$e'_\alpha = e_\beta V^\beta_\alpha, \tag{3.2}$$

we arrive at a new matrix for operator T, a matrix connected with $\| D^\beta_\alpha \|$ via the similarity transformation

$$D'^\beta_\alpha = (V^{-1}DV)^\beta_\alpha. \tag{3.3}$$

Indeed, from (3.2) we find that

$$e_\alpha = (V^{-1})^\beta_\alpha e'_\beta,$$

and then

$$\mathrm{T}e'_\alpha = \mathrm{T}e_\beta V^\beta_\alpha = e_\gamma D^\gamma_\beta V^\beta_\alpha = e'_\delta (V^{-1})^\delta_\gamma D^\gamma_\beta V^\beta_\alpha = e'_\delta (V^{-1}DV)^\delta_\alpha.$$

Two representations, D and $\mathrm{V}^{-1}\mathrm{DV}$, related through a similarity transformation are said to be equivalent.

Thus, a representation of the Lorentz group can be considered as being a set of linear operators acting in the finite-dimensional space

of the components of field functions $u^A(x)$ and not only in the space of 4-vectors x^μ. An important idea in the theory of the representations of groups is that of reducibility. A representation of a group G in a space R is said to be irreducible if in R there are no subspaces invariant with respect to this representation except the null space and R itself, otherwise it is called reducible. For instance, an example of a reducible representation is the $(n + m)$-dimensional representation whose matrices are of the diagonal-block type:

$$D = \begin{pmatrix} \overbrace{D_1}^{n} & \overbrace{0}^{m} \\ 0 & D_2 \end{pmatrix} \begin{matrix} \} n \\ \} m \end{matrix}, \tag{3.4}$$

where D_1 and D_2 are n-by-n and m-by-m representation matrices, respectively.

Separating the representation space into invariant subspaces, we obtain irreducible representatons of the group, whereby studying a reducible representation results in studying all the irreducible representations of this group. By studying the finite-dimensional representations of the Poincaré group we can obtain the possible types of field functions $u(x)$ and the laws that govern their transformation

$$u^A(x) \to u'^A(x') = D^A_B u^B(x), \tag{3.5}$$

when the coordinates are transformed thus:

$$x^\mu \to x'^\mu = A^\mu_\nu x^\nu + a^\mu. \tag{3.6}$$

As an example of the above-said, let us consider the tensor representation

$$u'^{\mu\nu\cdots\rho}(x') = A^\mu_{\alpha_1} A^\nu_{\alpha_2} \cdots A^\rho_{\alpha_N} u^{\alpha_1\alpha_2\cdots\alpha_N}(x),$$

where N Lorentz transformation matrices A are multiplied by $u^{\alpha_1\alpha_2\ldots\alpha_N}(x)$, the components of a contravariant tensor of rank N that are even with respect to coordinate inversion (see above). The corresponding pseudotensors change sign under reflection of an odd number of spatial coordinate axes.

Particular cases of a tensor representation are

(1) the scalar (pseudoscalar) representation

$$u'(x') = u(x), \tag{3.7}$$

(2) the contravariant vector (pseudovector) representation

$$u'^\nu(x') = A^\nu_\mu u^\mu(x), \tag{3.8}$$

(3) the covariant vector (pseudovector) representation

$$u^*_\nu(x') = (A^{-1})^\mu_\nu u_\mu(x). \tag{3.9}$$

A special case is the spinor representation of the Lorentz group. This representation, together with the spinor representation of the rotation group, will be obtained later.

The Poincaré and $O(3)$ groups discussed above belong to what is known as Lie groups. A Lie group is characterized by the fact that all the transformations result in the most general transformation that depends analytically on the r independent (or essential) parameters a^i, for instance,

$$x'^{\nu} = f^{\nu}(x^1, \ldots, x^n, a^1, \ldots, a^r).$$

A more general definition of a Lie group consists in the following.

Let us consider a continuous, or topological, group, that is, a group such that (1) in the set of the group elements we can introduce the concepts of proximity (an infinitely small neighborhood), convergence, etc., and (2) from the smallness in the variation in one multiplicand of the group follows the smallness of the variation of the product.

Let us assume that to each element A of a given topological group we can assign, in a one-to-one manner, a set of essential parameters $a = (a^1, \ldots, a^r)$; these sets form the so-called group, or parametric, space. Such parametrization is fixed to within an arbitrary N-times differentiable transformation $a' = f(a)$.

The one-to-one correspondence

$$A \leftrightarrow (a^1, \ldots, a^r), \quad B \leftrightarrow (b^1, \ldots, b^r), \quad AB = C, \quad C \leftrightarrow (c^1, \ldots, c^r) \tag{3.10}$$

implies the group composition law

$$c^i = \varphi^i(a, b), \tag{3.11}$$

where φ^i is an analytic function of a and b.

Such a topological group constitutes an r-parameter Lie group.

Suppose that in a certain representation of a Lie group there are operators T_a depending on parameter a. Then the one-to-one correspondence (3.10) leads to the following restrictions on φ^i:

(1) the associativity

$$T_a(T_bT_c) = (T_aT_b)T_c,$$

whence

$$\varphi[a, \varphi(b, c)] = \varphi[\varphi(a, b), c]; \tag{3.12}$$

(2) the existence of a unit element:

$$T_{a_0}T_a = T_aT_{a_0} = T_a,$$

where $T_{a_0} = 1$ is the unit operator, whence

$$a = \varphi(a, a_0) = \varphi(a_0, a), \tag{3.13}$$

and in what follows we put $a_0 = 0$;

(3) the existence of an inverse element:

$$T_{\bar{a}}T_a = T_aT_{\bar{a}} = 1,$$

where $T_{\bar{a}} = T_a^{-1}$, i.e.

$$a_0 = \varphi(a, \bar{a}) = \varphi(\bar{a}, a) = 0, \tag{3.14}$$

which implies that we can find $\bar{a}$ if $\det(\partial\varphi^i/\partial a^j) \neq 0$.

Infinitely small (or infinitesimal) transformations occupy a special place in the theory of Lie groups. Suppose we have a representation of a Lie group in which the elements are the operators T_a that are analytic in parameters a, that is, $T_a = A(a)$. Let us introduce the operators

$$X_i = \left(\frac{\partial A}{\partial a^i}\right)_{a=0} \quad (i = 1, 2, \ldots, r), \tag{3.15}$$

which are known as the infinitesimal operators of the group, or generators of the group. Then in the neighborhood of the unit element for small values of a we obtain the following expression for a group element in the linear approximation:

$$A = I + a^iX_i. \tag{3.16}$$

As we recall from the definition of a Lie group, parametrization is defined to within the transformation

$$a^i \to a'^i = f^i(a), \quad \det(\partial f^i/\partial a^j) \neq 0.$$

Then for the new parameters a'^i we obtain the generators

$$X_i' = \left(\frac{\partial A}{\partial a'^i}\right)_0 = \left(\frac{\partial A}{\partial a_j}\right)_0 \left(\frac{\partial a^j}{\partial a'^i}\right)_0 = X_j \left(\frac{\partial a^j}{\partial a'^i}\right)_0, \tag{3.17}$$

where the indices 0 mean that the derivatives are taken at $a = 0$ and $a' = 0$, respectively.

We will now derive a remarkable relationship connecting the generators of a Lie group. Suppose that $A = A(a)$ and $B = B(b)$ are operators belonging to a certain representation of a Lie group. We set up the product $C = BAB^{-1}$. If the group is noncommutative, that is, $BA \neq AB$, then $C \neq A$. For a commutative, or Abelian, group, for which the order in which the factors occur in a product is unimportant, that is, $BA = AB$, we get, obviously, $C = A$. The product C has corresponding to it a parameter

$$c = \varphi[b, \varphi(a, \bar{b})]. \tag{3.18}$$

For infinitely small transformations,

$$A = I + a^iX_i, \quad B = I + b^iX_i. \tag{3.19}$$

We write the product as follows:

$$C = BAB^{-1} = I + a^i X_i - a^i b^j (X_i X_j - X_j X_i) = I + a^i X_i', \tag{3.20}$$

where

$$X_i' = X_i - b^j (X_i X_j - X_j X_i). \tag{3.21}$$

On the other hand,

$$C = I + c^i X_i, \tag{3.22}$$

and according to (3.17),

$$X_i' = \frac{\partial c^k}{\partial a^i} X_k. \tag{3.23}$$

Let us now find parameter c^k. Allowing for (3.18), we have, to within second-order terms,

$$\varphi^k (a, \bar{b}) = a^k + \bar{b}^k + f^k_{ji} a^j \bar{b}^i + f'^k_{ji} a^j a^i + f''^k_{ji} \bar{b}^j \bar{b}^i.$$

Since $\varphi (a, 0) = a$ and $\varphi (0, \bar{b}) = \bar{b}$, the coefficients f' and f'' vanish, and therefore

$$c^k = a^k + f^k_{ji} (b^j a^i - b^j b^i - a^j b^i), \tag{3.24}$$

where we have allowed for the fact that $\bar{b} = -b$ near the origin o coordinates.

If we differentiate (3.24) with respect to a^i, we get

$$\frac{\partial c^k}{\partial a^i} = \delta^k_i + (f^k_{ji} - f^k_{ij}) b^j = \delta^k_i + c^k_{ij} b^j, \tag{3.25}$$

with

$$c^k_{ij} = f^k_{ji} - f^k_{ij}$$

being the antisymmetric combinations ($c^k_{ij} = -c^k_{ji}$) known as structure constants.

If we now substitute (3.25) into (3.23), we get

$$X_i' = X_i - X_k c^k_{ij} b^j. \tag{3.26}$$

Comparing (3.26) with (3.21), we arrive at an important relation:

$$X_i X_j - X_j X_i = c^k_{ij} X_k, \tag{3.27}$$

or, in brief form,

$$[X_i, X_j] = c^k_{ij} X_k. \tag{3.28}$$

The expression

$$[X_i, X_j] \equiv X_i X_j - X_j X_i \tag{3.29}$$

is called a commutator and it satisfies the obvious antisymmetry condition

$$[X_i, X_j] = -[X_j, X_i], \tag{3.30}$$

and the Jacobi identity

$$[[X_i, X_j], X_k] + [[X_k, X_i], X_j] + [[X_j, X_k], X_i] = 0. \tag{3.31}$$

The Jacobi identity can easily be verified on the basis of the definition of a commutator. It provides an obvious restriction on the structure constants.

Formula (3.28) does not depend on the representation. This is clear if only from the fact that the explicit form of a representation was not used when we derived (3.28). This implies that the structure constants, too, are independent of the representation. These constants are the individual characteristics of a Lie group, whose generators satisfy the commutation relations (3.28). We will see that knowing the operators of infinitesimal transformations makes it possible to reconstruct the entire group of finite transformations.

Note, however, that the explicit form of the commutation relations and the structure constants depends on the choice of the parameters, which, as we know, are not determined uniquely. The structure constants c_{ij}^k are tensors with respect to linear transformations a^i, as follows from the definition of these constants.

The linear space formed by the generators of a Lie group is known as the Lie algebra of that group. In this algebra the product of elements is given in the form of commutators (3.29) and the Jacobi identity (3.31) is the associativity condition.

Knowing the Lie algebra of a certain Lie group, that is, knowing the generators and the structure constants, is sufficient for building the Lie group, that is, for finding all the finite elements that form the group. To verify this statement, let us assume that in G there is a one-parameter subgroup*, that is, a subgroup whose elements T_t are additively dependent on a single parameter t:

$$T_{t_1} T_{t_2} = T_{t_1+t_2}. \tag{3.32}$$

This means that in the neighborhood of the origin in the parameter space we have specified a curve

$$a^i = a^i(t), \quad a^i(0) = 0 \tag{3.33}$$

and the law of multiplication has the form

$$a^i(t_1 + t_2) = \varphi^i(a(t_1), \quad a(t_2)). \tag{3.34}$$

* A subgroup is a group whose elements constitute a part of the elements of group G, that is, elements plus a multiplication law and a unit element.

The last relationship can be considered as the functional equation for determining (3.33).

Let us consider a representation of the one-parameter subgroup in the parameter space

$$T_\tau a^i(t) = a^i(t + \tau). \tag{3.35}$$

It is clear that in this representation the generator has the form $X_t = d/dt$. We take the derivatives of both sides of (3.35) with respect to parameter τ and use the fact that $d/dt = d/d\tau$, which follows from the additivity of the group. As a result we get

$$\frac{dT_\tau}{d\tau} a^i(t) = \frac{d}{dt}[T_\tau a^i(t)], \tag{3.36}$$

or

$$\frac{dT_\tau}{d\tau} a^i(t) = X_t T_\tau a^i(t). \tag{3.37}$$

This leads to the following operator equation

$$\frac{dT_\tau}{d\tau} = X_t T_\tau. \tag{3.38}$$

The solution of this equation with the initial condition $T_0 = I$ can be written in the form of a series:

$$T_\tau = I + \tau X_t + \frac{\tau^2}{2} X_t^2 + \ldots = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} (X_\tau)^n, \tag{3.39}$$

which has an infinite radius of convergence in τ. In brief form we can write this series as an operator exponential function:

$$T_\tau = \exp(\tau X_\tau), \tag{3.40}$$

which clearly satisfies the additivity condition (3.32). This solution, obtained in representation (3.35), must hold in an arbitrary representation as well, as follows from the aforesaid. Both (3.39) and (3.40) enable finding the operator of an arbitrary finite transformation in a given one-parameter subgroup knowing the appropriate generator.

Using the fact that the choice of the parameters of a group is not unique, we now specify the parametrization of a group in such a manner that the law of composition becomes additive

$$\varphi^i(a, b) = a^i + b^i$$

for each $i = 1, 2, \ldots, r$. Such a system of parameters is said to be canonical.

Assigning to one parameter, say a^i, arbitrary admissible values $a^i = \tau$ and setting all the other parameters zero, $a^k = 0$ $(k \neq i)$,

we arrive at a one-parameter subgroup. Its operators can be represented, in accord with (3.40), as

$$T_{a^i} = \exp(a^i X_i). \tag{3.41}$$

This formula does not contain summation over i, and the generator is $X_i = \partial/\partial a^i$.

In the same way we can construct the one-parameter subgroups corresponding to the other parameters a^k. The total number of such subgroups is equal to the number of parameters r.

In group theory it is proved in general form that any element of a Lie group is either an element of a one-parameter subgroup or can be represented in the form of a product of such elements (Lie's theorem; see [6]). Thus, knowing the operators of infinitely small transformations (generators), one can reconstruct the entire group. We will not prove this theorem in its most general form. Instead we will illustrate the theorem by constructing the group from one-parameter subgroups using the concrete examples of groups of coordinate transformations: the groups of translations and three-dimensional rotations and the Lorentz group.

Let us first write out the generators explicitly, in the coordinate representation. Take the coordinate transformation

$$x^\mu \to x'^\mu = f^\mu(x, a) = T_a x^\mu,$$

depending on r parameters a^i. Near the identity transformation $x^\mu = f^\mu(x, 0)$ we have

$$x'^\mu = x^\mu + dx^\mu = x^\mu + \left[\frac{\partial f^\mu(x, a)}{\partial a^i}\right]_{a=0} a^i. \tag{3.42}$$

Let us consider an arbitrary scalar function of coordinates, $\psi(x)$. Under a coordinate transformation $x \to Tx$ this function does not change:

$$\psi(x) \to \psi'(x') = \psi'(Tx) = \psi(x).$$

From this follows the law of change of the form of the function that compensates for coordinate transformation:

$$\psi'(x) = \psi(T^{-1}x).$$

For infinitesimal transformations we have

$$\psi'(x) = \psi(x) + dx^\mu \frac{\partial \psi(x)}{\partial x^\mu},$$

where dx^μ corresponds to a transformation that is the inverse to (3.42)

$$dx^\mu = -\left[\frac{\partial f^\mu}{\partial a^i}\right]_{a=0} a^i,$$

so that

$$\psi'(x) = \psi(x) - a^i \left[\frac{\partial f^\mu}{\partial a^i}\right]_{a=0} \frac{\partial}{\partial x^\mu} \psi(x).$$

Thus, we can write the generators of the transformation as follows:

$$X_i = -\left[\frac{\partial f^\mu(x, a)}{\partial a^i}\right]_{a=0} \frac{\partial}{\partial x^\mu}. \tag{3.43}$$

3.2 Examples of Continuous Transformation Groups

Using the above-described concepts, we will now show how the basic groups used in field theory can be constructed.

3.2.1. Translation group. Let us consider the coordinate transformations

$$x^\mu \to x'^\mu = f^\mu(x, a) = x^\mu + a^\mu, \tag{3.44}$$

where a^μ is a 4-vector in Minkowski's space-time and does not depend on x^μ. These are continuous transformations, and they constitute a Lie group with four parameters $a = (a^0, a^1, a^2, a^3)$, the translation group. The parameter, or group, space in the given case coincides with the x-space, or Minkowski's space-time.

In what follows we consider, as is common practice in the literature, the coordinate transformations associated with the change in position of points in space, which correspond to the change in position of physical objects, while the base vectors will remain unchanged. In the case with (3.44), the change in coordinates is due to translation of all objects by a constant vector a^μ. Substituting the transformation law (3.44) into the general formula (3.43), we obtain

$$X_\mu = -\left[\frac{\partial f^\nu(x, a)}{\partial a^\mu}\right]_{a=0} \frac{\partial}{\partial x^\nu} = -\delta_\mu^\nu \frac{\partial}{\partial x^\nu}.$$

Thus, the derivatives

$$X_\mu = -\partial/\partial x^\mu \quad (\mu = 0, 1, 2, 3) \tag{3.45}$$

are the generators of the translation. Since the parameters a^μ are additive, each can be used to construct a subgroup of the type

$$T_\mu = \exp(-a^\mu \partial/\partial x^\mu) \quad (\mu = 0, 1, 2, 3), \tag{3.46}$$

where summation over μ is not carried out.

Obviously, when the translation operator (3.46) is applied to a function $\psi = \psi(x)$, the result is the transformation of component

x^μ, while all other components x^ν ($\nu \neq \mu$) remain unchanged:

$$T_\mu \psi(\ldots, x^\mu, \ldots) = \left[1 - a^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{2}(a^\mu)^2 \frac{\partial^2}{\partial (x^\mu)^2} + \ldots\right] \psi(\ldots, x^\mu, \ldots) = \psi(\ldots, x^\mu - a^\mu, \ldots).$$

The algebra of the group generators is commutative:

$$[X_\mu, X_\nu] = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} = 0,$$

whereby the operator of the general transformation T_a, that is, translation along vector a, can be written in the form of the product of four consecutive translations of the (3.46) type:

$$T_a = T_0 T_1 T_2 T_3, \tag{3.47}$$

and the order of these translations is arbitrary. This makes it possible to rewrite (3.47) as

$$T_a = \exp(-a^\mu \partial/\partial x^\mu), \tag{3.48}$$

where we must sum over the repeated index μ from 0 to 3.

Note that the spatial part of the operator (3.48), $T_a = \exp(-\mathbf{a}\cdot\nabla)$, is the operator of spatial translations. The corresponding generator, $-\nabla$, is related to the quantum mechanical hermitian momentum operator, $\mathbf{P} = -i\hbar\nabla$.

3.2.2. Rotation-in-a-plane group. A rotation in the xy-plane about the z axis by an angle φ leads to the following transformation of the coordinates:

$$\begin{aligned} x' &= x\cos\varphi - y\sin\varphi, \\ y' &= x\sin\varphi + y\cos\varphi. \end{aligned} \tag{3.49}$$

This yields the following matrix representation of the group operators T_φ, which depend on a single parameter φ:

$$T_\varphi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad T_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{3.50}$$

The range of variation of parameter φ fixes the group space, which is the circle $0 < \varphi \leqslant 2\pi$. The infinitesimal transformation can be obtained if we put $\varphi = \delta\varphi \to 0$:

$$T_{\delta\varphi} = I + \begin{pmatrix} 0 & -\delta\varphi \\ \delta\varphi & 0 \end{pmatrix} = I + \delta\varphi X_\varphi,$$

where in the matrix representation the generator is

$$X_\varphi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.51}$$

In the coordinate representation, via the general formula (3.43), we obtain

$$X_\varphi = -\left(\frac{\partial x'}{\partial \varphi}\right)_{\varphi=0} \frac{\partial}{\partial x} - \left(\frac{\partial y'}{\partial \varphi}\right)_{\varphi=0} \frac{\partial}{\partial y} = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}. \quad (3.52)$$

The generator of the transformation of a function $\psi = \psi(\varphi)$ that depends on angle φ can also be found via the same rule as the generator of the transformation of a function of coordinates, or (3.43):

$$X_\varphi = -\left[\frac{\partial(\varphi+\varphi^1)}{\partial \varphi^1}\right]_{\varphi^1=0} \frac{\partial}{\partial \varphi} = -\frac{\partial}{\partial \varphi}. \quad (3.53)$$

Then for a finite rotation through an angle $\Delta\varphi$ we get

$$T_{\Delta\varphi} = \exp(\Delta\varphi X_\varphi) = \exp\left(-\Delta\varphi \frac{\partial}{\partial \varphi}\right). \quad (3.54)$$

The matrix of operator T_φ (3.50) is orthogonal since rotation does not change the length of a vector, $x^2 + y^2 = x'^2 + y'^2$. From this it follows that

$$T_\varphi^T T_\varphi = I, \quad (3.55)$$

that is, the transposed matrix is the inverse of T_φ. This is clear also from (3.50), since the transition to an inverse transformation, $\varphi \to -\varphi$, leads to a transposed matrix.

For infinitely small rotations we can write (3.55) in the linear approximation

$$(I + \delta\varphi X_\varphi^T)(I + \delta\varphi X_\varphi) = I,$$

whence

$$X_\varphi^T = -X_\varphi$$

(the antisymmetry of the generator matrix). Note that the generator X_φ is related to the quantum mechanical hermitian operator of the projection of the angular momentum on the z axis (in units of Planck's constant $\hbar$) thus:

$$L_z = -i\frac{\partial}{\partial \varphi} = +iX_\varphi.$$

Then the operator of finite rotations can be written in the form

$$T_{\Delta\varphi} = \exp(-i\Delta\varphi L_z).$$

3.2.3. Three-dimensional rotation group $O(3)$. The general element of the group, the operator T_φ, depends on three parameters, the components of the vector $\varphi = \varphi\mathbf{n} = (\varphi^1, \varphi^2, \varphi^3)$, where $\mathbf{n}$ is a unit vector ($|\mathbf{n}| = 1$) and $|\varphi| < \pi$, that is, $-\pi < \varphi < \pi$. All rotations are confined to a sphere of radius π. This is the parameter space, and it is clear that this space is finite. Any two points

in opposition on the surface of the sphere correspond to identity transformations, a rotation through an angle of 180°. Such points are therefore assumed to be identical.

Let us isolate the one-parameter subgroups. Suppose that $\varphi = (\varphi^1, 0, 0)$, which represents a rotation about the x axis by an angle of φ^1. The coordinates are transformed according to the following law:

$$\begin{aligned} x' &= x, \\ y' &= y\cos\varphi^1 - z\sin\varphi^1, \\ z' &= y\sin\varphi^1 + z\cos\varphi^1. \end{aligned} \tag{3.56}$$

This results in the following matrix of operator T_1:

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi^1 & -\sin\varphi^1 \\ 0 & \sin\varphi^1 & \cos\varphi^1 \end{pmatrix}, \tag{3.57}$$

and the generator in the matrix representation is

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.58}$$

while in the coordinate representation we have, according to (3.43)

$$X_1 = -\left(\frac{\partial \mathbf{r}'}{\partial \varphi^1}\right)_{\varphi^1=0} \nabla = -y\frac{\partial}{\partial z} + z\frac{\partial}{\partial y}. \tag{3.59}$$

For rotations about the y axis we take the vector $\varphi = (0, \varphi^2, 0)$ and similarly find the corresponding operator of finite transformations:

$$T_2 = \begin{pmatrix} \cos\varphi^2 & 0 & \sin\varphi^2 \\ 0 & 1 & 0 \\ -\sin\varphi^2 & 0 & \cos\varphi^2 \end{pmatrix}, \tag{3.60}$$

with the generator in the matrix representation being

$$X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.61}$$

and in the coordinate representation,

$$X_2 = -\left(\frac{\partial \mathbf{r}'}{\partial \varphi^2}\right)_{\varphi^2=0} \nabla = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}. \tag{3.62}$$

Finally, for rotations about the z axis, that is, for $\varphi = (0, 0, \varphi^3)$, we have

$$T_3 = \begin{pmatrix} \cos\varphi^3 & -\sin\varphi^3 & 0 \\ \sin\varphi^3 & \cos\varphi^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.63}$$

with the generator in the matrix representation being

$$X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.64}$$

and in the coordinate representation,

$$X_3 = -\left(\frac{\partial \mathbf{r}'}{\partial \varphi^3}\right)\nabla = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}. \tag{3.65}$$

In the matrix-vector representation, the generator matrices (3.58), (3.61), and (3.64) can be represented in compact form in the following manner:

$$(X_i)_{kl} = \varepsilon_{kil}, \tag{3.66}$$

where ε_{kil} is the totally antisymmetric unit tensor ($\varepsilon_{123} = 1$). Then the rotation of a vector $\mathbf{r}$ through a small angle $\delta\varphi$ about an arbitrary axis $\mathbf{n} = \delta\varphi/|\,\delta\varphi\,|$ yields*

$$r_k' = r_k + (X_i \delta\varphi^i)_{kl} r_l = r_k + \varepsilon_{kil}\delta\varphi^i r_l,$$

or Euler's formula

$$\mathbf{r}' = \mathbf{r} + \delta\varphi \times \mathbf{r}. \tag{3.67}$$

The rotation generators we have just obtained are related to the quantum mechanical hermitian operator of angular momentum in the following manner:

$$L_i = +iX_i \quad (i = 1, 2, 3).$$

In the coordinate representation, we can find the operator of angular momentum by employing (3.59), (3.62), and (3.65). The result is $\mathbf{L} = -i\mathbf{r} \times \nabla$; while in the matrix representation (3.66) we have $(L_i)_{kl} = i\varepsilon_{kil}$.

We can show directly that the generators satisfy the following commutation relations:

$$[X_1, X_2] = X_3, \quad [X_3, X_1] = X_2, \quad [X_2, X_3] = X_1, \tag{3.68}$$

* Here and in what follows the Cartesian coordinates of vector $\mathbf{r}$ are denoted thus: $r_1 = x$, $r_2 = y$, and $r_3 = z$.

or in general form

$$[X_i, X_k] = \varepsilon_{ikj} X_j. \tag{3.69}$$

Correspondingly, for the hermitian generators, the components of the angular momentum, we have

$$[L_i, L_k] = i\varepsilon_{ikj} L_j. \tag{3.70}$$

Note that the commutation relations (3.69) and (3.70) are valid in an arbitrary representation of the O (3) group (the only requirement here is that the group is parametrized by angles φ^i ($i = 1, 2, 3$)).

Let us consider the one-parameter subgroup of rotations about a fixed axis n through an angle φ. The generator of this subgroup is defined by (3.17):

$$X_\varphi = X_i \frac{\partial \varphi^i}{\partial \varphi} = X_i \frac{\partial}{\partial \varphi} (\varphi n^i) = X_i n^i,$$

where n^i are the coordinates of the unit vector n, the direction cosines of the rotation axis.

For the general element of the subgroup we have

$$T_\varphi = \exp(\varphi X_\varphi) = \exp(\varphi^1 X_1 + \varphi^2 X_2 + \varphi^3 X_3). \tag{3.71}$$

Note that by virtue of the noncommutativity of the group, a property that follows from the commutation relations (3.69), the operator T_φ given by (3.71) cannot be written in the form of a product of exponentials of the form

$$\exp(\varphi^3 X_3) \exp(\varphi^2 X_2) \exp(\varphi^1 X_1).$$

This means that we cannot arrive at the resulting rotation (3.71) about the n axis through an angle φ by sequential rotations about the axes *1*, *2*, and *3* through angles φ^1, φ^2, and φ^3. This is possible only for infinitely small rotations in the linear approximation (3.67). For finite φ^i, the result of sequential rotations about different axes depends on the order in which these rotations are performed. This manifests itself in the noncommutativity of rotations, mathematically expressed by (3.69).

To describe an arbitrary rotation, it is customary practice to introduce the so-called Euler angles, known from the mechanics of a rigid body. An arbitrary rotation then can be written as a sequence of rotations through Euler angles. Let us consider an arbitrary rotation of the base depicted in Figure 3.1. This rotation can be described as the product of three sequential rotations:

$$T_{\varphi^3 \varphi^2 \varphi^1} = T'_{\varphi''} T'_{\varphi^2} T_{\varphi^3}. \tag{3.72}$$

The operators on the right-hand side correspond to the following rotations: T_{φ^3}, the rotation about axis *3* through an angle φ^3, T'_{φ^2},

the rotation about axis l through an angle φ^1; and $T'_{\varphi'^3}$, the rotation about axis $3'$ through an angle φ'^3. The rotations T'_{φ^1} and $T'_{\varphi'^3}$ occur about the auxiliary axis l and the new axis $3'$. Therefore, these rotations correspond to new operators, and these are labeled by primes. They can be expressed in terms of rotations about the initial axes *1* and *3*:

$$T'_{\varphi^1} = T_{\varphi^3} T_{\varphi^1} T_{\varphi^3}^{-1},$$
$$T'_{\varphi'^3} = (T'_{\varphi^1} T_{\varphi^3})\, T_{\varphi'^3}\, (T'_{\varphi^1} T_{\varphi^3})^{-1}. \tag{3.73}$$

Substituting (3.73) into (3.72), we get

$$T_{\varphi^3\varphi^1\varphi'^3} = T_{\varphi^3} T_{\varphi^1} T_{\varphi'^3}. \tag{3.74}$$

We have thus proved that an arbitrary element of the $O(3)$ group parametrized by angles φ^3, φ^1, and φ'^3 (Euler angles) can be represented in the form of a product of elements of one-parameter groups, T_1 and T_3. The reader can see that to construct an arbitrary rotation it suffices to employ only two subgroups of O (3). This is the result of the commutation relations (3.69), which connect three generators of the group. It is clear that instead of the *1* and *3* axes we could have chosen any other pair: *2* and *3* or *3* and *1*.

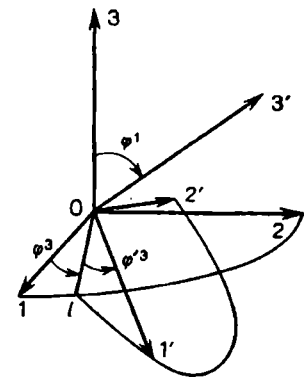

Figure 3.1 The Euler angles.

3.2.4. The proper Lorentz group $L_+^\uparrow$. This is the group of pseudorotations in the four-dimensional pseudo-Euclidean space that do not change the length of a 4-vector:

$$(x^0)^2 - \mathbf{x}^2 = (x'^0)^2 - \mathbf{x}'^2 = \text{inv}. \tag{3.75}$$

Let us study the one-parameter subgroups of $L_+^\uparrow$. A pseudorotation in the 01-plane results in a transformation $x^0, x^1 \to x'^0, x'^1$, while the coordinates x^2 and x^3 remain unchanged. Let us employ (2.8) to write this transformation explicitly:

$$x'^0 = x^0 \cosh \tilde{\varphi}^1 + x^1 \sinh \tilde{\varphi}^1,$$
$$x'^1 = x^0 \sinh \tilde{\varphi}^1 + x^1 \cosh \tilde{\varphi}^1, \tag{3.76}$$
$$x'^2 = x^2, \quad x'^3 = x^3,$$

where parameter $\tilde{\varphi}^1$ is related to the velocity v of the intrinsic reference frame along axis *1* through the formula $\tanh \tilde{\varphi}^1 = v/c$. Clearly, (3.76) satisfies (3.75). The matrix of transformation (3.76), which

fixes the representation of group $L_+^\uparrow$ in Minkowski's space-time, has the form

$$T_{\tilde{\varphi}^1} = \| A_\nu^\mu \| = \begin{pmatrix} \cosh\tilde{\varphi}^1 & \sinh\tilde{\varphi}^1 & 0 & 0 \\ \sinh\tilde{\varphi}^1 & \cosh\tilde{\varphi}^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.77}$$

As can easily be seen, the parameter $\tilde{\varphi}^1$ is additive,

$$T_{\tilde{\varphi}'^1} T_{\tilde{\varphi}^1} = T_{\tilde{\varphi}'^1 + \tilde{\varphi}^1},$$

and therefore using it instead of velocity v is preferable. Such parameters are usually called velocity parameters.

The appropriate infinitesimal parameter can be obtained if we send $\tilde{\varphi}^1$ to zero. Then

$$T_{\tilde{\varphi}^1} = I + \tilde{\varphi}^1 \tilde{X}_1,$$

where the group generator $\tilde{X}_1 = (\partial/\partial\tilde{\varphi}^1)\, T_{\tilde{\varphi}^1}\,|_{\tilde{\varphi}^1=0}$ is

$$\tilde{X}_1 = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right). \tag{3.78}$$

Here the zero 2-by-2 blocks in the matrix are denoted by 0, for the sake of brevity.

The other subgroups of pseudorotations can be obtained in a similar manner. We will not write the expressions for finite transformations, since they all have the same form (3.76). We simply write the infinitesimal pseudorotations in the 02-plane,

$$T_{\tilde{\varphi}^2} = I + \tilde{\varphi}^2 \tilde{X}_2,\ \tilde{X}_2 = \left(\begin{array}{cc|cc} 0 & & 1 & 0 \\ & & 0 & 0 \\ \hline 1 & 0 & & \\ 0 & 0 & & 0 \end{array}\right), \tag{3.79}$$

and in the 03-plane,

$$T_{\tilde{\varphi}^3} = I + \tilde{\varphi}^3 \tilde{X}_3,\ \tilde{X}_3 = \left(\begin{array}{cc|cc} 0 & & 0 & 1 \\ & & 0 & 0 \\ \hline 0 & 0 & & \\ 1 & 0 & & 0 \end{array}\right). \tag{3.80}$$

The other transformations in $L_+^\uparrow$ are common rotations, that is, rotations that do not include the temporal coordinate x^0. They belong to the $O(3)$ group, which is a subgroup of $L_+^\uparrow$. The corresponding generators of the one-parameter subgroups with parameters φ^1, φ^2, and φ^3 can be obtained if we turn to (3.58), (3.61), and (3.64) and add a zero row and a zero column, precisely,

$$X_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right), \quad X_2 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right),$$

$$X_3 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{3.81}$$

Thus, the proper Lorentz group $L_+^\uparrow$ is specified by six parameters: $\tilde{\varphi}^1$, $\tilde{\varphi}^2$, $\tilde{\varphi}^3$ and φ^1, φ^2, φ^3. The six generators of the group, X_i and $\tilde{X}_i$ ($i = 1, 2, 3$), can be denoted in a more symmetric way if instead of one index $i = 1, 2, 3$, we introduce a pair of indices α and β varying as $\alpha, \beta = 0, 1, 2, 3$, namely,

$$X_{ik} = \varepsilon_{ikj} X_j, \quad X_{i0} = \tilde{X}_i \quad (i, \ k = 1, 2, 3), \tag{3.82}$$

or $X_{12} = X_3$, etc., and $X_{10} = \tilde{X}_1$, etc. Then, as the reader can easily see, the matrices (3.78), (3.79), (3.80), and (3.81) will be represented in a more compact form as follows:

$$(X_{\alpha\beta})^\mu_\nu = \delta^\mu_\alpha g_{\nu\beta} - \delta^\mu_\beta g_{\nu\alpha}, \tag{3.83}$$

where δ^μ_α is the Kronecker delta ($\mu, \alpha = 0, 1, 2, 3$), and $g_{\nu\beta}$ is the metric tensor. Lowering the indices, we obtain

$$(X_{\alpha\beta})_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}. \tag{3.84}$$

This compact notation of the matrix representation of the generators of the Lorentz group will prove extremely convenient in our future discussions. Note the fact that matrix (3.84) is antisymmetric,

$$(X_{\alpha\beta})_{\mu\nu} = -(X_{\alpha\beta})_{\nu\mu}, \tag{3.85}$$

with $X_{\alpha\beta} = -X_{\beta\alpha}$.

By direct calculation we can establish the following value of the commutator of the pseudorotation generators:

$$[\tilde{X}_1, \tilde{X}_2] = -X_3,$$

and similarly for the other commutators involving $\tilde{X}_i$. In general,

$$[\tilde{X}_i, \tilde{X}_j] = -\varepsilon_{ijk}X_k. \tag{3.86}$$

In the same manner we can obtain the other commutation relations:

$$[X_i, \tilde{X}_j] = \varepsilon_{ijk}\tilde{X}_k, \tag{3.87}$$

$$[X_i, X_j] = \varepsilon_{ijk}X_k. \tag{3.88}$$

Obviously, (3.88) coincides with the commutation relation (3.69) for the generators of the O (3) group, a subgroup of $L_+^\uparrow$.

An arbitrary infinitesimal transformation of the proper Lorentz group $L_+^\uparrow$ has the form

$$x'^\mu = x^\mu + [\delta\varphi^i X_i + \delta\tilde{\varphi}^i\tilde{X}_i]^\mu_\nu x^\nu.$$

Similarly, we can show, by a method we employed when we were studying the O (3) group, that in the Lorentz group an arbitrary transformation can be constructed from one-parameter subgroups, say

$$A = T'_{\varphi'}T_{\tilde{\varphi}^i}T_\varphi, \tag{3.89}$$

where T_φ and $T'_{\varphi'}$ are the matrices of the group of three-dimensional rotations, O (3).

4. Integrals of the Field Equations

In the first three sections of this chapter we considered the Lagrange and Hamilton methods for systems with an infinitude of degrees of freedom, methods that make it possible to obtain field equations in general form using the variational principle as a basis. We also considered the main properties of transformations in Minkowski's space-time of special relativity, properties that lead to the concept of a Lorentz group. In this connection we discussed some results of the theory of groups and especially Lie groups (continuous transformations). Combining these two approaches, the group-theoretic approach and the Lagrange-Hamilton approach, Emmy Noether discovered her famous theorem, a result published in 1918 in Göttingen. The theorem states that under certain conditions there exists a set of integrals of motion (dynamical invariants) that characterize a field (or a sys-

tem of fields). More than that, the theorem provides a method for constructing the complete set of such integrals for any systems of fields with a given Lagrangian invariant with respect to a certain group of continuous transformations.

4.1 Noether's Theorem

Let us consider a continuous (topological) group of coordinate transformations: $x \to x' = f(x, a)$. Suppose that we have a system of field functions $u^A = u^A(x)$ that realize the linear representation of this group:

$$x \to x', \quad u^A(x) \to u'^A(x'), \quad A = 1, 2, \ldots, N. \tag{4.1}$$

A function (or functional) I for which

$$I(x, u(x), \partial u/\partial x, \ldots) = I(x', u'(x'), \partial u'/\partial x', \ldots)$$

is said to be an invariant of the group. In the action integral we can write

$$S = \int d^4x \mathcal{L}(x, u, \partial u/\partial x), \tag{4.2}$$

where integration is carried out over any real range x and we assume that $\mathcal{L}$ depends only on derivatives whose order is no higher than the first, $\partial u/\partial x$. Thus, S is action with variable limits of integration, and $\mathcal{L}$ is the density of the Lagrangian. The invariance of functional S means that

$$S = \int d^4x \mathcal{L}(x, u, \partial u/\partial x) = \int d^4x' \mathcal{L}(x', u', \partial u'/\partial x') \tag{4.3}$$

for any transformations of the (4.1) type.

Let us consider the variation of S:

$$\delta S = \int d^4x' \mathcal{L}(x', u'(x'), \partial u'/\partial x') - \int d^4x \mathcal{L}(x, u(x), \partial u/\partial x). \tag{4.4}$$

First we separate the portion of the variation that is linked with the variation in coordinates, $x \to x' = x + \delta x$, and the variation caused by a change in the form of $u^A(x)$,

$$u^A(x) \to u'^A(x) = u^A(x) + \bar{\delta} u^A.$$

The variation of $u^A(x)$ that is not due to the variation in coordinates is called local. The total variation

$$\delta u^A = u'^A(x') - u^A(x) = \bar{\delta} u^A(x) + \frac{\partial u^A}{\partial x^\nu} \delta x^\nu \tag{4.5}$$

is said to be the essential variation of u^A. The variation in the Lagrangian density can be written in the form

$$\delta\mathscr{L} = \bar{\delta}\mathscr{L} + \frac{\partial \mathscr{L}}{\partial x^\nu}\delta x^\nu, \tag{4.6}$$

where $\bar{\delta}\mathscr{L}$ is the variation in $\mathscr{L}$ due to variations in the form of function u, and the procedure for finding the derivatives of $\mathscr{L}$ presupposes both the explicit and implicit dependences of $\mathscr{L}$ on x. The elements of a 4-volume in the new and old coordinates are related through the formula

$$d^4x' = \det \left\| \frac{\partial x'^\mu}{\partial x^\nu} \right\| d^4x.$$

Let us estimate the magnitude of the Jacobian to within linear terms. We use the identity

$$\det A = e^{\mathrm{Tr}\ln A}, \tag{4.7}$$

which is valid for every matrix A. In the linear approximation for matrix $A = \| \partial x'^\mu/\partial x^\nu \|$ we have

$$A = \| \delta^\mu_\nu + \partial\delta x^\mu/\partial x^\nu \|,$$

$$(\ln A)^\mu_\nu = \ln(\delta^\mu_\nu + \partial\delta x^\mu/\partial x^\nu) = \partial\delta x^\mu/\partial x^\nu.$$

Then identity (4.7) yields (in the same approximation)

$$\det A = \exp(\partial\delta x^\mu/\partial x^\mu) = 1 + \partial\delta x^\mu/\partial x^\mu.$$

This gives us the sought formula:

$$d^4x' = \left(1 + \frac{\partial\delta x^\nu}{\partial x^\nu}\right) d^4x.$$

On the whole, for the action integral we get

$$\delta S = \int d^4x \left\{ \bar{\delta}\mathscr{L} + \frac{\partial}{\partial x^\nu}(\mathscr{L}\delta x^\nu) \right\}. \tag{4.8}$$

The variation of the form of $\mathscr{L}$ is

$$\bar{\delta}\mathscr{L} = \frac{\partial \mathscr{L}}{\partial u^A}\bar{\delta}u^A + \frac{\partial \mathscr{L}}{\partial u^A_{,\nu}}\bar{\delta}u^A_{,\nu}. \tag{4.9}$$

Since the variation $\bar{\delta}u^A$ is local, we can write

$$\bar{\delta}u^A_{,\nu} = \frac{\partial}{\partial x^\nu}\bar{\delta}u^A. \tag{4.10}$$

Substituting (4.9) into (4.8), allowing for (4.10), and performing simple identity transformations, we can rewrite δS thus:

$$\delta S = \int d^4x \left\{ \frac{\delta S}{\delta u^A} \bar{\delta} u^A + \frac{\partial}{\partial x^\nu} \left(\mathcal{L} \delta x^\nu + \frac{\partial \mathcal{L}}{\partial u^A_{,\nu}} \bar{\delta} u^A \right) \right\}, \qquad (4.11)$$

where the functional derivative is

$$\frac{\delta S}{\delta u^A} \equiv \frac{\partial \mathcal{L}}{\partial u^A} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathcal{L}}{\partial u^A_{,\nu}}.$$

Suppose action is invariant, that is, condition (4.3) is satisfied. Then $\delta S = 0$ and, in view of the fact that the domain of integration is arbitrary, (4.11) yields

$$\frac{\delta S}{\delta u^A} \bar{\delta} u^A + \frac{\partial}{\partial x^\nu} \left(\mathcal{L} \delta x^\nu + \frac{\partial \mathcal{L}}{\partial u^A_{,\nu}} \bar{\delta} u^A \right) = 0. \qquad (4.12)$$

The reader will recall that the transformations (4.1) constitute a group. Therefore,

$$\delta x^\mu = (X_l)^\mu_\nu x^\nu \delta a^l,$$
$$\delta u^A(x) = (X_l)^A_B u^B(x)\, \delta a^l, \quad l = 1, 2, \ldots, r. \qquad (4.13)$$

Here $(X_l)^\mu_\nu$ and $(X_l)^A_B$ are the generators of the group in the x- and the u-representation. Since all the a^l are independent, the coefficients of each variation δa^l in (4.12) must vanish independently. We have therefore arrived at the theorem formulated by Noether as follows.

If the functional S is invariant with respect to an r-parameter Lie group, then r linearly independent combinations of the functional derivatives $\delta S/\delta u^A$ turn into divergences. Actually, most often we have to do with a variational problem, that is, the field functions u^A satisfy the Lagrange equations

$$\delta S/\delta u^A = 0 \quad (A = 1, 2, \ldots, N).$$

Therefore, (4.12) implies that

$$\frac{\partial}{\partial x^\nu} \left[\mathcal{L} \delta x^\nu + \frac{\partial \mathcal{L}}{\partial u^A_{,\nu}} \bar{\delta} u^A \right] = 0.$$

Taking into account the expressions for variations, (4.5) and (4.13), we find that

$$\frac{\partial}{\partial x^\nu} \left\{ \mathcal{L} (X_l)^\nu_\mu x^\mu + \frac{\partial \mathcal{L}}{\partial u^A_{,\nu}} [(X_l)^A_B u^B - u^A_{,\lambda} (X_l)^\lambda_\mu x^\mu] \right\} \delta a^l = 0.$$

In view of the fact that the variations of the parameters, δa^i, are independent, from the last equation we get r continuity equations

$$\frac{\partial}{\partial x^\nu} J_i^\nu = 0 \quad (i = 1, 2, \ldots, r) \tag{4.14}$$

for the Noether currents defined thus:

$$J_i^\nu = -\mathscr{L} (X_i)_\mu^\nu x^\mu + \frac{\partial \mathscr{L}}{\partial u^A_{,\nu}} [u^A_{,\lambda} (X_i)_\mu^\lambda x^\mu - (X_i)_B^A u^B]. \tag{4.15}$$

Equation (4.14) can also be written in integral form:

$$\int d^4x \frac{\partial}{\partial x^\nu} J_i^\nu = 0, \tag{4.16}$$

where integration is carried out over an arbitrary volume of Minkowski's space-time. Next we employ the divergence theorem and rewrite (4.16) in the form of a surface integral over the boundary of the volume:

$$\int d^4x \frac{\partial}{\partial x^\nu} J_i^\nu = \int_\sigma d\sigma_\nu J_i^\nu = 0. \tag{4.17}$$

Here $d\sigma_\nu$ is the projection of the element of hypersurface σ encompassing the 4-volume onto the 3-plane that is perpendicular to the x^ν axis, namely, $d\sigma_0 = dx^1\, dx^2\, dx^3$, $d\sigma_1 = dx^0\, dx^2\, dx^3$, etc. Let us assume that the fields and the currents J_i^ν ($\nu = 1, 2, 3$) vanish at spatial infinity, that is,

$$J_i^\nu |_{|r| \to \infty} \to 0 \quad (\nu = 1, 2, 3).$$

Then, selecting the volume of integration in such a way that in spacelike directions it extends to infinity, we get, from (4.17),

$$\int_{t=t_1} J_i^0 d^3x = \int_{t=t_2} J_i^0 d^3x. \tag{4.18}$$

In these integrals, $d^3x = dx^1 dx^2 dx^3 = d\sigma_0$ are elements of 3-volumes, that is, in essence the area elements of the hyperplanes corresponding to the constant values $t = t_1$ and $t = t_2$, respectively. If we introduce the so-called Noether charges,

$$Q_i = \int J_i^0 d^3x,$$

we will find, as a consequence of (4.18), that they are time independent: $dQ_i/dt = 0$ ($i = 1, 2, \ldots, r$).

We therefore arrive at another formulation of Noether's theorem: if the action integral is an invariant of an r-parameter Lie group,

then, in view of the field equations, we have r integrals of motion that do not vary with time (dynamical invariants, Noether charges):

$$Q_i = \int J_i^0 \, d^3x = \text{constant} \quad (i = 1, 2, \ldots, r). \tag{4.19}$$

Equation (4.19) is not written in covariant form since time is explicitly isolated. We can write covariant equations if while integrating in (4.17) we restrict the volume of integration not by hyperplanes t = constant but by spacelike surfaces* in the timelike directions, assuming that in spacelike directions the volume extends to infinity, with the currents J_i^ν falling off to zero in these directions. Then instead of (4.19) we get

$$Q_i = \int_\sigma J_i^\nu \, d\sigma_\nu = \text{constant}. \tag{4.20}$$

Here the charges are defined as integrals over spacelike surfaces, and the reader can easily see that they do not depend on the spacelike surface σ.

Note that Noether currents are not uniquely defined since we can always add to them divergences of any antisymmetric tensors $f_i^{\mu\nu} = -f_i^{\nu\mu}$. Indeed, in this case the continuity equations (4.14) are not violated, since for an antisymmetric tensor we have

$$\frac{\partial^2}{\partial x^\nu \, \partial x^\mu} f_i^{\nu\mu} \equiv 0.$$

More than that, the charges Q_i (which are conserved) do not change their form since

$$\int d^3x \frac{\partial f_i^{0\mu}}{\partial x^\mu} = \int d^3x \frac{\partial}{\partial x^k} f_i^{0k} = \int ds_k f_i^{0k} = 0 \quad (k = 1, 2, 3),$$

where the fields and f^{0k} are assumed to vanish on the surface encompassing the three-dimensional volume (the last integral is taken over this surface).

4.2 The Energy-Momentum Tensor. The Angular Momentum of the Field

Let us consider the integrals of motion that emerge as a result of the requirement that action be invariant under transformations belonging to the translation group and the proper Lorentz group.

* A spacelike surface possesses the following property: a normal to any point on this surface is a timelike vector, with the result that any two points on the surface are separated by a spacelike interval.

Under translation transformations, $x'^{\nu} = x^{\nu} + \delta a^{\nu}$, the field functions do not transform: $u'^{A}(x') = u^{A}(x)$. Since

$$\delta x^{\nu} = (X_{\alpha})_{\mu}{}^{\nu} x^{\mu} \delta a^{\alpha} = \delta a^{\nu},$$

we can write

$$(X_{\alpha})_{\mu}{}^{\nu} x^{\mu} = \delta_{\alpha}{}^{\nu}. \tag{4.21}$$

At the same time $\delta u^{A} = 0$, and therefore

$$(X_{\alpha})_{B}{}^{A} = 0. \tag{4.22}$$

Substituting (4.21) and (4.22) into the definition (4.15) of the Noether current, we arrive at a new quantity,

$$T_{\alpha}{}^{\nu} = \frac{\partial \mathscr{L}}{\partial u^{A}_{,\nu}} u^{A}_{,\alpha} - \mathscr{L} \delta^{\nu}_{\alpha}, \tag{4.23}$$

known as the canonical energy-momentum tensor of the system (the field). Note that the upper index is placed somewhat to the right of the lower index so as to leave room for raising the lower index when we go over from the mixed tensor (4.23) to the contravariant tensor

$$T^{\alpha\nu} = \frac{\partial \mathscr{L}}{\partial u^{A}_{,\nu}} u^{A,\alpha} - \mathscr{L} g^{\alpha\nu}. \tag{4.24}$$

By Noether's theorem, the invariance of action under translations, $\delta S = 0$, implies the continuity equation

$$\frac{\partial T^{\alpha\nu}}{\partial x^{\nu}} = 0 \quad (\alpha = 0, 1, 2, 3). \tag{4.25}$$

The integral conservation law has the form

$$\int T^{\alpha 0} d^{3}x = \text{costant} = P^{\alpha} \quad (\alpha = 0, 1, 2, 3), \tag{4.26}$$

where integration is carried out over the entire volume occupied by the field. The P^{α} are the components of the 4-momentum of the field. Indeed, let us write

$$T^{00} = \frac{\partial \mathscr{L}}{\partial u^{A}_{,t}} u^{A}_{,t} - \mathscr{L} = \mathscr{H}. \tag{4.27}$$

As we can see, the T^{00} component of the energy-momentum tensor coincides with the Hamiltonian density of the field, $\mathscr{H}$. Therefore, the volume integral

$$P^{0} = \int d^{3}x \mathscr{H} = \int d^{3}x T^{00} \tag{4.28}$$

determines the total energy of the field. In view of the fact that the P^α are covariant quantities, the other components are those of the total momentum of the field:

$$P^k = \int d^3x T^{k0} = \int d^3x \frac{\partial \mathcal{L}}{\partial u^A_{,t}} u^{A,k} \quad (k = 1, 2, 3). \tag{4.29}$$

Thus, in view of the homogeneity of space-time, which property is expressed by the condition $\delta S = 0$ as $x \to x + \delta a$, we arrive at the conservation law for the total 4-momentum of the field: $P^\alpha = \text{constant}$. It must be noted at this point that when there are several interacting fields of varying nature, the 4-momentum of the entire system is conserved. In this case, the indices A, in the definition (4.23) of the energy-momentum tensor, over which indices summation is performed, stand not only for the field components but for the type of the field as well.

Let us now consider the proper Lorentz transformations. Using the notation by which the parameters of the generators of a group are included in two indices, (3.82), we can write the variations of the coordinates and field functions as

$$\begin{aligned} \delta x^\nu &= (X_{\alpha\beta})^\nu_\mu x^\mu \delta a^{\alpha\beta}, \\ \delta u^A &= (X_{\alpha\beta})^A_B u^B \delta a^{\alpha\beta}, \end{aligned} \tag{4.30}$$

where $a^{12} = \varphi^3$, etc., and $a^{10} = \tilde{\varphi}^1$, etc. For the spatial part of the Noether current, which is not involved in the transformations of the field functions, the definition (4.15) yields

$$L^\nu_{\alpha\beta} = u^A_{,\sigma} (X_{\alpha\beta})^\sigma_\mu x^\mu \frac{\partial \mathcal{L}}{\partial u^A_{,\nu}} - (X_{\alpha\beta})^\nu_\mu x^\mu \mathcal{L} = (X_{\alpha\beta})^\sigma_\mu x^\mu T^\nu_\sigma,$$

where $T_\sigma{}^\nu$ is the energy-momentum tensor defined in (4.23). Recalling the explicit form of the generator of the Lorentz group,

$$(X_{\alpha\beta})^\sigma_\mu = \delta^\sigma_\alpha g_{\beta\mu} - \delta^\sigma_\beta g_{\alpha\mu}, \tag{4.31}$$

we find that

$$L^\nu_{\alpha\beta} = T^\nu_\alpha x_\beta - T^\nu_\beta x_\alpha. \tag{4.32}$$

The quantities $L^\nu{}_{\alpha\beta}$ (4.32) form a tensor of rank three, the tensor of the angular momentum of the field, which is antisymmetric in the lower indices:

$$L^\nu{}_{\alpha\beta} = -L^\nu{}_{\beta\alpha}.$$

Let us now consider the field part of the Noether current, the part that is involved in the transformations of the field functions:

$$S^{\nu}_{\alpha\beta} = -(X_{\alpha\beta})^{A}_{B} u^{B} \frac{\partial \mathscr{L}}{\partial u^{A}_{,\nu}}. \tag{4.33}$$

These quantities form a tensor of rank three, the spin, or intrinsic, angular momentum tensor of the field, which characterizes the transformation properties of the field with respect to spatial-temporal rotations (polarization properties).

To calculate $S^{\nu}{}_{\alpha\beta}$ we must know the representation of the generators $X_{\alpha\beta}$ in the space of the field functions u^{A}. If the field is scalar (one-component), that is, $u'(x') = u(x)$, then $(X_{\alpha\beta})^{A}_{B} \equiv 0$ and, hence,

$$S^{\nu}_{\alpha\beta} \equiv 0.$$

If the field is a vector field, that is, $u^{\nu}(x)$ $(\nu = 0, 1, 2, 3)$, the generators $(X_{\alpha\beta})^{\nu}{}_{\mu}$ are defined by (4.31). For a spinor field we must know the spinor representation of the Lorentz group (see below).

The continuity equation

$$\frac{\partial}{\partial x^{\nu}} (L^{\nu}_{\alpha\beta} + S^{\nu}_{\alpha\beta}) = 0$$

describes the conservation law for the total angular momentum of the field, or the sum

$$L^{\nu}_{\alpha\beta} + S^{\nu}_{\alpha\beta}.$$

If the field is a scalar field, then $S^{\nu}{}_{\alpha\beta} = 0$ and, hence,

$$\frac{\partial}{\partial x^{\nu}} L^{\nu}_{\alpha\beta} = 0,$$

whence

$$\frac{\partial}{\partial x^{\nu}} L^{\nu[\alpha\beta]} = \frac{\partial}{\partial x^{\nu}} (T^{\alpha\nu} x^{\beta} - T^{\beta\nu} x^{\alpha}) = T^{\alpha\beta} - T^{\beta\alpha} = 0,$$

that is,

$$T^{\alpha\beta} = T^{\beta\alpha}.$$

Thus, in the case of a scalar field the canonical energy-momentum tensor is symmetric. In the general case, however, this is not so, but if we add the divergence of an antisymmetric tensor $f^{[\nu\beta]\alpha}$,

$$T^{\nu\alpha} \to T^{\nu\alpha} + \frac{\partial}{\partial x^{\beta}} f^{[\nu\beta]\alpha},$$

the resulting tensor will be symmetric. The new tensor is known as the metric energy-momentum tensor.

Integrating the components $L^0{}_{\alpha\beta}$ and $S^0{}_{\alpha\beta}$ over the 3-space, we get

$$L_{\alpha\beta} = \int d^3x L^0_{\beta\alpha}, \quad S_{\alpha\beta} = \int d^3x S^0_{\beta\alpha}, \tag{4.34}$$

which are, respectively, the orbital angular momentum tensor and the spin, or intrinsic, angular momentum tensor of the field, both being antisymmetric:

$$L_{\alpha\beta} = -L_{\beta\alpha}, \quad S_{\alpha\beta} = -S_{\beta\alpha}.$$

In the general case of an arbitrary field their sum is conserved:

$$\frac{d}{dt}(L_{\alpha\beta} + S_{\alpha\beta}) = 0.$$

The spatial components L_{ik},

$$L_{23} = L_1, \quad L_{31} = L_2, \quad L_{12} = L_3, \tag{4.35}$$

form an axial vector of the field angular momentum:

$$L_i = \frac{1}{2}\varepsilon_{ikl}L_{kl} \quad (i, k, l = 1, 2, 3). \tag{4.36}$$

Accordingly, for the intrinsic angular momentum of the field we get

$$S_i = \frac{1}{2}\varepsilon_{ikl}S_{kl}, \tag{4.37}$$

which is the spin vector of the field. Thus, the isotropy of the field, that is, the fact that $\delta S = 0$ under spatial rotations, leads to the conservation of the total angular momentum of all the interacting fields in a system:

$$L_i + S_i = \text{constant} \quad (i = 1, 2, 3). \tag{4.38}$$

5. Spinor Representation of the $O(3)$ Group

We have already familiarized ourselves with the tensor representations of the Lorentz group and the rotation group. These are single-valued representations, since we can establish a one-to-one correspondence between the elements of an abstract group and the matrices in a definite tensor representation of this group. But two-valued representations of the O (3) group also have physical meaning. These are known as spinor representations.

5.1 The SU (2) Group

Let us consider the two-dimensional linear complex space of vectors (spinors)

$$c = c^1 e_1 + c^2 e_2 \tag{5.1}$$

with complex-valued base vectors e_1 and e_2. We denote the components of vector c, or spinor, by a column and the components of a base vector by a row:

$$\| c^i \| = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix}, \quad \| e_i \| = (e_1, e_2). \tag{5.2}$$

Let us define the scalar product of two vectors, c and d, in the following manner:

$$(c, d) = c^{*1}d^1 + c^{*2}d^2 = c^+d, \tag{5.3}$$

where

$$c^+ = (c^{*1}, c^{*2}). \tag{5.4}$$

Let us consider a linear transformation in this space,

$$c' = Uc. \tag{5.5}$$

We assume that this transformation is unitary, that is, preserves the invariance of the scalar product:

$$c' = Uc, \quad d' = Ud, \quad (c, d) = (c', d') = \text{inv}. \tag{5.6}$$

Let us write condition (5.6) explicitly:

$$(c', d') = (Uc)^+Ud = c^+U^+Ud = c^+d.$$

From this immediately follows

$$U^+ = U^{-1}. \tag{5.7}$$

Such matrices are known as unitary. Next we assume that the transformation matrices are unimodular, or

$$\det U = 1. \tag{5.8}$$

The matrix of a general linear transformation has the form

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{5.9}$$

where α, β, γ, and δ are arbitrary complex numbers. Then the linear transformation of vector components can be written as follows:

$$\begin{aligned} c'^1 &= \alpha c^1 + \beta c^2, \\ c'^2 &= \gamma c^1 + \delta c^2. \end{aligned} \tag{5.10}$$

Accordingly, for the base vectors we can write out the transformation $e' = eU$ explicitly:

$$\begin{aligned} e'_1 &= \alpha e_1 + \gamma e_2, \\ e'_2 &= \beta e_1 + \delta e_2. \end{aligned} \tag{5.11}$$

The requirement (5.8) that U be unimodular leads to the following condition:

$$\alpha\delta - \beta\gamma = 1. \tag{5.12}$$

It can easily be verified that the inverse of U, provided (5.12) is satisfied, has the form

$$U^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Since

$$U^{+} = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix},$$

the unitarity condition (5.7) yields

$$\alpha^* = \delta, \quad \gamma^* = -\beta.$$

Thus, the unitary unimodular transformation has the form

$$\begin{aligned} c'^1 &= \alpha c^1 + \beta c^2, \\ c'^2 &= -\beta^* c^1 + \alpha^* c^2, \end{aligned} \tag{5.13}$$

and its matrix

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \tag{5.14}$$

depends on three real parameters, since the four real numbers α_1, α_2, β_1, β_2, where $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$, obey the single condition

$$\alpha\alpha^* + \beta\beta^* = 1. \tag{5.15}$$

We can easily see that the set of matrices of the (5.14) type augmented by condition (5.15) forms a group. Indeed, the product of any two unitary unimodular matrices is a matrix of the same type, each matrix U has an inverse, U^{-1}, since $\det U = 1 \neq 0$, and the unit matrix I belongs to the same set. This group of special (unimodular) unitary matrices of the transformation of the linear complex two-dimensional space is denoted by SU (2). The group space of SU (2) is the surface of the unit sphere in four-dimensional real space:

$$|\alpha|^2 + |\beta|^2 = \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1.$$

Let us investigate the corollaries that follow from the unimodularity and unitarity of the transformations that constitute this group. For two vectors c and d we get

$$c' = Uc, \quad d' = Ud.$$

Let det U = 1. We construct the quadratic combination

$$c'^1d'^2 - c'^2d'^1 = (\alpha c^1 + \beta c^2)(\gamma d^1 + \delta d^2) - (\gamma c^1 + \delta c^2)(\alpha d^1 + \beta d^2)$$
$$= (\alpha\delta - \beta\gamma)(c^1d^2 - c^2d^1),$$

and in view of (5.12) we have

$$c'^1d'^2 - c'^2d'^1 = c^1d^2 - c^2d^1 = \text{inv}. \tag{5.16}$$

From this it follows that the components c^2 and $-c^1$ are transformed covariantly with respect to the components of the basis, that is,

$$(c^2, -c^1) \sim (e_1, e_2), \tag{5.17}$$

and form a covariant vector.

Next, in view of the unitary condition we have

$$c^{*1}c^1 + c^{*2}c^2 = \text{inv},$$

and therefore c^{*1} and c^{*2} are transformed covariantly with respect to e_1 and e_2, that is, also form a covariant vector

$$(c^{*1}, c^{*2}) \sim (e_1, e_2). \tag{5.18}$$

Finally, the components c^{*2} and $-c^{*1}$ are transformed under unitary transformations contravariantly with respect to c^2 and $-c^1$ and, hence, covariantly with respect to c^1 and c^2, that is, form a contravariant vector

$$\begin{pmatrix} c^{*2} \\ -c^{*1} \end{pmatrix} \sim \begin{pmatrix} c^1 \\ c^2 \end{pmatrix}. \tag{5.19}$$

5.2 Linear Representations of $SU(2)$

Let us consider the linear space whose base vectors are homogeneous products of vectors e_i of order $N = 1, 2, 3, 4, \ldots$:

$$e_1^N, e_1^{N-1}e_2, \ldots, e_2^N. \tag{5.20}$$

When $N = 1$, we have the initial two-dimensional space. In general, the linear space of vectors

$$c^0e_1^N + c^1e_1^{N-1}e_2 + \ldots + c^Ne_2^N \tag{5.21}$$

has a dimension of $N + 1$. Under linear transformations of the basis of the initial space $e \to eU$, (5.11), the base vectors (5.20) are also transformed linearly, since the result,

$$e_1^{N-s}e_2^s \to (\alpha e_1 + \gamma e_2)^{N-s}(\beta e_1 + \delta e_2)^s, \tag{5.22}$$

forms a linear combination of homogeneous terms of the (5.20) type. Thus, we have arrived at linear representations of the SU (2) group.

Commonly, N is replaced by the quantity $j = N/2 = 1/2, 1, 3/2, \ldots$, with the result that the dimension of the representation space is $N + 1 = 2j + 1$. For $N = 1$ we have $j = 1/2$; this is the representation of $SU(2)$ in itself, or the spinor representation. For $N = 2$ ($j = 1$) the dimension of the representation space is $2j + 1 = 3$; this is the vector representation of the $SU(2)$ group.

Let us consider the three-dimensional vector representation ($j = 1$) in greater detail. The representation space consists of vectors

$$c = c^0 e_1^2 + c^1 e_1 e_2 + c^2 e_2^2 = \xi^0 \varepsilon_0 + \xi^1 \varepsilon_1 + \xi^2 \varepsilon_2, \qquad (5.23)$$

where we have introduced a new basis $\varepsilon_0 = e_1^2/\sqrt{2}$, $\varepsilon_1 = e_1 e_2$, $\varepsilon_2 = e_2^2/\sqrt{2}$ and new components $\xi^0 = \sqrt{2} c^0$, $\xi^1 = c^1$, $\xi^2 = \sqrt{2} c^2$.

Vector (5.23) can be represented in the form of a product of two vectors $d = d^1 e_1 + d^2 e_2$ and $g = g^1 e_1 + g^2 e_2$. Therefore, the quadratic combination

$$(c^1)^2 - 4c^0 c^2 \sim (d^1 g^2 + d^2 g^1)^2 - 4 d^1 g^1 d^2 g^2 = (d^1 g^2 - d^2 g^1)^2 = \text{inv}, \qquad (5.24)$$

in accord with (5.16), is an invariant of the vector representation of the $SU(2)$ group. The following quantity is also an invariant, in view of the unitarity of the spinor representation of the $SU(2)$ group, (5.6):

$$|\xi^0|^2 + |\xi^1|^2 + |\xi^2|^2 \sim 2(|d^1|^2 + |d^2|^2) = \text{inv}. \qquad (5.25)$$

Thus, the vector representation (5.23) of the $SU(2)$ group is also unitary. Let us consider the transformation $\xi^0, \xi^1, \xi^2 \to x, y, z$:

$$x = \frac{1}{\sqrt{2}}(\xi^2 - \xi^0), \quad y = -\frac{i}{\sqrt{2}}(\xi^2 + \xi^0), \quad z = \xi^1. \qquad (5.26)$$

This is simply a unitary similarity transformation in the representation space:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = V \begin{pmatrix} \xi^0 \\ \xi^1 \\ \xi^2 \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}. \qquad (5.27)$$

The inverse matrix is $V^{-1} = V^+$. The transformation (5.27) preserves the scalar product

$$|\xi^0|^2 + |\xi^1|^2 + |\xi^2|^2 = x^2 + y^2 + z^2. \qquad (5.28)$$

If the matrix of transformation (5.22) in the space ξ^0, ξ^1, ξ^2 is denoted by D_1, then an equivalent transformation in the space x,

y, z, is carried out by a matrix related to D_1 in the following manner (see Section 3):

$$D_1' = VD_1V^+. \tag{5.29}$$

In view of (5.24) and (5.25), the sum of the squares of the new components,

$$x^2 + y^2 + z^2 = (c^1)^2 - 4c^0c^2 = |\xi^0|^2 + |\xi^1|^2 + |\xi^2|^2, \tag{5.30}$$

is also an invariant of the group. Thus, the transformation D_1', the vector representation of $SU(2)$, is a rotation. What are the conditions that ensure that this rotation is real? Allowing for the representation (5.23) of c in the form of the product of d and g and going over to the components (5.26), we get

$$x \sim d^2g^2 - d^1g^1, \quad y \sim -i(d^1g^1 + d^2g^2), \quad z \sim d^1g^2 + d^2g^1. \tag{5.31}$$

In view of (5.19), for the transformations belonging to $SU(2)$ group we have $g^1 \sim d^{*2}$ and $g^2 \sim -d^{*1}$, whereby (5.31) is replaced by the following:

$$x \sim -(d^2d^{*1} + d^1d^{*2}), \; y \sim i(d^2d^{*1} - d^1d^{*2}),$$
$$z \sim -d^1d^{*1} + d^2d^{*2}. \tag{5.32}$$

We see that x, y, z are transformed as real combinations of the components of spinor d. Hence, the transformation coefficients must also be real, that is, x, y, z undergo real rotations belonging to the $O(3)$ group under transformations D_1'.

Let us find the explicit form of the matrix of real rotations D_1'. Applying directly the law of transformations of base vectors e_1 and e_2,

$$e_1' = \alpha e_1 - \beta^* e_2,$$
$$e_2' = \beta e_2 + \alpha^* e_2,$$

we get

$$(\varepsilon_0', \; \varepsilon_1', \; \varepsilon_2') = (\varepsilon_0, \; \varepsilon_1, \; \varepsilon_2) D_1,$$

where

$$D_1 = \begin{pmatrix} \alpha^2 & \sqrt{2}\alpha\beta & \beta^2 \\ -\sqrt{2}\alpha\beta^* & \alpha\alpha^* - \beta\beta^* & \sqrt{2}\alpha^*\beta \\ \beta^{*2} & -\sqrt{2}\alpha^*\beta^* & \alpha^{*2} \end{pmatrix}. \tag{5.33}$$

If we then carry out the similarity transformation (5.29) using the matrix (5.27), we finally get

$$D'_1 = \begin{pmatrix} \frac{\alpha^2-\beta^{*2}+\alpha^{*2}-\beta^2}{2} & \frac{\alpha^2-\beta^{*2}-\alpha^{*2}+\beta^2}{2i} & -(\alpha\beta+\alpha^*\beta^*) \\ \frac{\alpha^{*2}+\beta^2-\alpha^2-\beta^{*2}}{2i} & \frac{\alpha^2+\beta^2+\alpha^{*2}+\beta^{*2}}{2} & \frac{\alpha\beta-\alpha^*\beta^*}{i} \\ \alpha\beta^*+\alpha^*\beta & \frac{\alpha\beta^*-\alpha^*\beta}{i} & \alpha\alpha^*-\beta\beta^* \end{pmatrix}. \quad (5.34)$$

As we see, all the coefficients of (5.34) are real. Let us now show that each real rotation is contained in the representation D'_1, that is, we wish to build the operator of an arbitrary rotation, the general element of the O (3) group.

Let us consider two matrices belonging to the SU (2) group of the type

$$A(\alpha_1) = \begin{pmatrix} \cos\alpha_1 & -i\sin\alpha_1 \\ -i\sin\alpha_1 & \cos\alpha_1 \end{pmatrix}, \quad C(\alpha_3) = \begin{pmatrix} e^{-i\alpha_3} & 0 \\ 0 & e^{i\alpha_3} \end{pmatrix}. \quad (5.35)$$

In representation D'_1, from the general expression we obtain

$$A(\alpha_1) \to T_{2\alpha_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\alpha_1 & -\sin 2\alpha_1 \\ 0 & \sin 2\alpha_1 & \cos 2\alpha_1 \end{pmatrix}, \quad (5.36)$$

$$C(\alpha_3) \to T_{2\alpha_3} = \begin{pmatrix} \cos 2\alpha_3 & -\sin 2\alpha_3 & 0 \\ \sin 2\alpha_3 & \cos 2\alpha_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (5.37)$$

The matrix (5.36) coincides with the matrix (3.57) representing rotations about the first axis through the angle $\varphi^1 = 2\alpha_1$, while the matrix (5.37) coincides with the matrix (3.63) representing rotations about the third axis through the angle $\varphi^3 = 2\alpha_3$. In Section 3 it was demonstrated that an arbitrary rotation belonging to the O (3) group can be constructed out of the above-mentioned rotations if we introduce the Euler angles φ^1, φ^3, φ'^3:

$$T_{\varphi'^3\varphi^3\varphi'^3} = T_{\varphi'^3} T_{\varphi^3} T_{\varphi'^3}. \quad (3.74)$$

Therefore, the general element of the SU (2) group parametrized by the Euler angles and having representation (3.74) in O (3) can be found in the following manner:

$$U = C\left(\tfrac{1}{2}\varphi^3\right) A\left(\tfrac{1}{2}\varphi^1\right) C\left(\tfrac{1}{2}\varphi'^3\right).$$

As a result we get

$$U_{\varphi^3\varphi^1\varphi'^3} = \begin{pmatrix} \exp\left(\frac{\varphi^3+\varphi'^3}{2i}\right)\cos\frac{\varphi^1}{2} & -i\exp\left(\frac{\varphi^3-\varphi'^3}{2i}\right)\sin\frac{\varphi^1}{2} \\ -i\exp\left(\frac{\varphi'^3-\varphi^3}{2i}\right)\sin\frac{\varphi^1}{2} & \exp\left(-\frac{\varphi^3+\varphi'^3}{2i}\right)\cos\frac{\varphi^1}{2} \end{pmatrix}. \tag{5.38}$$

However, the correspondence between $SU(2)$ and $O(3)$ is not one-to-one. Indeed, according to the general formula (5.34), the substitution of $-U$ for U, that is, $-\alpha$ and $-\beta$ for α and β, does not change the sign of D_1', in view of the quadratic dependence of D_1' on α and β. Therefore, the representation of D_1' via matrices U, or the spinor representation of $O(3) \to SU(2)$, is two-valued. Obviously, there are two different matrices in $SU(2)$ that correspond to the unit element in $O(3)$:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In a small neighborhood of I or $-I$ a one-to-one correspondence with the rotations from $O(3)$ can be established, that is, to each rotation D_1' through a small angle there will correspond a unique unitary transformation U close to I or $-I$. Under a continuous variation in the angle, the matrix U continuously changes with the rotation D_1'. However, when the rotation completes a full cycle $D_1' \to D_1'$, that is, $\varphi^1 \to \varphi^1 + 2\pi$ or $\varphi^3 \to \varphi^3 + 2\pi$ or $\varphi'^3 \to \varphi'^3 + 2\pi$, matrix U is replaced with $-U$. Thus, U is a continuous two-valued function of rotation D_1'.

Let us now find the generators of $SU(2)$ that realize the spinor representation of the corresponding generators of $O(3)$. According to the definition of generators, we get, from (5.35),

$$\begin{aligned} X_1 &= \left(\frac{\partial A(\varphi^1/2)}{\partial\varphi^1}\right)_{\varphi^1=0} = -\frac{i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ X_3 &= \left(\frac{\partial C(\varphi^3/2)}{\partial\varphi^3}\right)_{\varphi^3=0} = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{5.39}$$

Similarly to (5.36), we obtain the matrix of rotations about the second axis:

$$B\left(\frac{1}{2}\varphi^2\right) = \begin{pmatrix} \cos(\varphi^2/2) & -\sin(\varphi^2/2) \\ \sin(\varphi^2/2) & \cos(\varphi^2/2) \end{pmatrix}. \tag{5.40}$$

From this we obtain the generator

$$X_2 = -\frac{i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{5.41}$$

It is obvious that in the spinor representation the generators X_i satisfy the group commutation relation

$$[X_i, X_k] = \varepsilon_{ikj} X_j. \tag{3.69}$$

Let us now construct the hermitian generators by following the general rule (Section 3):

$$S_k = iX_k = \frac{1}{2}\sigma_k. \tag{5.42}$$

Here we have introduced the hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.43}$$

which are known as the Pauli matrices. They possess the following properties:

$$\begin{gathered} \sigma_1\sigma_2 = i\sigma_3, \ \sigma_2\sigma_3 = i\sigma_1, \ \sigma_3\sigma_1 = i\sigma_2, \\ \sigma_i\sigma_j + \sigma_j\sigma_i = 0, \ i \neq j, \\ (\sigma_i)^2 = \mathrm{I}, \end{gathered} \tag{5.44}$$

where I is the unit 2-by-2 matrix. In compact form these relations can be written as follows:

$$\sigma_i\sigma_k = \mathrm{I}\delta_{ik} + i\varepsilon_{ikj}\sigma_j. \tag{5.45}$$

Note that the Pauli matrices are traceless:

$$\mathrm{Tr}\ \sigma_i = 0, \tag{5.46}$$

which follows from the unimodularity of the transformation, det U = 1. The operators of finite rotations about the kth axis can, according to Section 3, be written in the form

$$U_k = \exp(\varphi^k X_k) = \exp(-i\varphi^k S_k) = \exp\left(-\frac{i}{2}\varphi^k\sigma_k\right). \tag{5.47}$$

In quantum mechanics, the hermitian operators (5.42) correspond to the components of the spin operators of particles with spin 1/2, while the spinors are vectors in the space in which these operators act. The eigenvectors of the operators S_k describe the quantum mechanical states with definite values of the projection of the spin on the kth axis, +1/2 or −1/2.

6. Intrinsic Symmetries of Fields. Gauge Transformations

Up till now we discussed only the transformations that involved coordinates (spatial coordinates, that is) and time in the case of the Lorentz group. The field functions realize the representation of the transformation group and transform according to a certain rule together with the coordinates. But coordinate transformations do not exhaust the possible transformations that the field functions can undergo. There exist what are known as intrinsic symmetries of physical objects, or elementary particles, that reflect the internal properties of the objects, properties that are in no way connected with possible displacements in space. These symmetries manifest themselves in transformations that do not involve coordinates; such transformations are known as gauge transformations.

6.1 Isotopic Rotations

Suppose that a field is described by complex-valued functions $u(x) = u_1 + iu_2$. Then, since the observables, such as, say, the Noether charges, which are integrals of the Lagrange equations, must be real-valued, they must depend only on the real quadratic combinations of field functions. This implies that the Lagrangian and, obviously, the action integral of this field must be invariant with respect to transformations of the type

$$\begin{aligned} u(x) &\to u'(x) = e^{i\alpha} u(x), \\ u^*(x) &\to u^{*\prime}(x) = e^{-i\alpha} u^*(x), \end{aligned} \tag{6.1}$$

where α is a continuous real parameter that is independent of coordinates x. The gauge transformations (6.1) do not involve the coordinates; they only change the form of the field functions, which are multiplied by a phase factor equal to unity in absolute value. Such transformations are unitary, that is, they do not change the norm of the field functions, and constitute a one-parameter unitary group, $U(1)$.

The infinitesimal transformations have the form

$$u'(x) = (1 + i\alpha)\, u(x), \quad u^{*\prime}(x) = (1 - i\alpha)\, u^*(x). \tag{6.2}$$

If we consider u' and $u^{*\prime}$ to be independent components of the field, we can write (6.2) in the form

$$\begin{pmatrix} u'(x) \\ u^{*\prime}(x) \end{pmatrix} = \left(1 + i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \alpha\right) \begin{pmatrix} u(x) \\ u^*(x) \end{pmatrix}, \tag{6.3}$$

from which follows the value of the generator:

$$X = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.4}$$

According to Noether's theorem, the Noether current that is connected with the transformations that do not involve the coordinates and is conserved is

$$J^\nu = -\frac{\partial \mathscr{L}}{\partial u_{,\nu}} Xu. \tag{6.5}$$

Here we have a matrix product of three factors, the first being a row and the last a column. Then, substituting (6.4) into (6.5), we get

$$j^\nu = -i\left(\frac{\partial \mathscr{L}}{\partial u_{,\nu}} u - \frac{\partial \mathscr{L}}{\partial u^*_{,\nu}} u^*\right). \tag{6.6}$$

This Noether current, which we have denoted by j^ν, is assumed to be the electromagnetic current associated with the complex-valued (and therefore charged) field u. By virtue of Noether's theorem, this current satisfies the continuity equation

$$\frac{\partial}{\partial x^\nu} j^\nu = 0, \tag{6.7}$$

and the associated charge, the electric charge, is conserved:

$$Q = \int d^3x j^0 = -i \int d^3x \left(\frac{\partial \mathscr{L}}{\partial u_{,t}} u - \frac{\partial \mathscr{L}}{\partial u^*_{,t}} u^*\right) = \text{constant}. \tag{6.8}$$

Thus, the gauge symmetry associated with the $U(1)$ group leads to electric charge conservation, and the electric charge is an additive quantity, in view of the fact that the one-parameter group is Abelian.

If instead of u and u^* we take the real part u_1 and the imaginary part u_2 of the field functions as the independent components of the field, then we have the following transformation law (instead of (6.1)):

$$\begin{aligned} u_1' &= u_1 \cos\alpha - u_2 \sin\alpha, \\ u_2' &= u_1 \sin\alpha + u_2 \cos\alpha. \end{aligned} \tag{6.9}$$

This transformation is nothing other than the rotation of the basis about the third axis through an angle α, the rotation belonging to the $O(1)$ group. Obviously, this rotation occurs not in the common space, the space where the position of material objects is specified, which is the configuration space, but in a special, intrinsic space of the field.

In similar manner one can specify the gauge symmetries connected with other additive charges, such as the hypercharge, the baryon number, and the lepton number (see below).

Transformation (6.9), as is known, is a subgroup of the group of three-dimensional orthogonal rotations, $O(3)$, whereby instead of (6.1) or (6.9) we can consider more general transformations in the intrinsic space of the fields, rotations $O(3)$ or (for complex-valued fields) the transformations of the $SU(2)$ group.

In the physics of elementary particles at present several intrinsic spaces of varying nature are known. One of these, which was introduced earlier than all other such spaces and which is related to the isotopic invariance of nuclear forces, is known as the isotopic space, with the corresponding orthogonal transformations called isotopic rotations. Since at present we need not know the particular nature of an intrinsic space, we will focus on isotopic rotations.

As an example, let us take the triplet of pions: π^-, π^0, and π^+.

The charged components of the triplet are described by the complex-valued functions $\pi^+ = \xi^0$ and $\pi^- = \xi^2$, while the neutral component is described by the real-valued function $\pi^0 = \xi^1$.

On the whole they form a three-dimensional representation of the $SU(2)$ group. Going over from the complex components ξ^0, ξ^1, and ξ^2 to the real components π^1, π^2, and π^3 via the well-known similarity transformation (5.27)

$$\pi = V\xi, \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{6.10}$$

we get

$$\pi^1 = \frac{1}{\sqrt{2}}(\xi^2 - \xi^0), \quad \pi^2 = \frac{1}{i\sqrt{2}}(\xi^0 + \xi^2), \quad \pi^3 = \xi^1, \tag{6.11}$$

and vice versa,

$$\xi^0 = -2^{-1/2}(\pi^1 - i\pi^2), \quad \xi^1 = \pi^3, \quad \xi^2 = 2^{-1/2}(\pi^1 + i\pi^2). \tag{6.12}$$

Suppose that

$$\xi^0 \to \xi'^0 = e^{-i\alpha}\xi^0, \quad \xi^1 \to \xi'^1 = \xi^1, \quad \xi^2 \to \xi'^2 = e^{i\alpha}\xi^2. \tag{6.13}$$

Then

$$\begin{aligned} \pi^1 \to \pi'^1 &= \pi^1 \cos\alpha - \pi^2 \sin\alpha, \\ \pi^2 \to \pi'^2 &= \pi^1 \sin\alpha + \pi^2 \cos\alpha, \\ \pi^3 \to \pi'^3 &= \pi^3. \end{aligned} \tag{6.14}$$

This transformation constitutes an isotopic rotation about the third axis and is the equivalent representation of complex-valued

vectors, (6.13). Similar transformations were considered in Section 3. The corresponding generator, as we found there, is

$$X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Returning to the complex representation (6.13), we obtain the matrix of the hermitian generator

$$I_3 = iX_3' = iV^+X_3V, \tag{6.15}$$

where V is the unitary matrix of the similarity transformation (6.10). In explicit form we have

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.16}$$

The other two generators can be found if we employ X_1 and X_2 defined in Section 3:

$$I_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad I_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{6.17}$$

The hermitian matrices $I_i = I_i^+$ $(i = 1, 2, 3)$ form the isospin operator in the vector representation of the isotopic rotation group SU (2). They obey the common commutation relations for angular momentum operators:

$$[I_i, I_j] = ie_{ijk}I_k. \tag{6.18}$$

The eigenvalues of the I_3 matrix are

$$I_3 = -1, 0, 1. \tag{6.19}$$

The charge Q of the particles, the components of the triplet π^-, π^0, π^+, can be associated with these eigenvalues:

$$Q = I_3. \tag{6.20}$$

Noether's charge of the isovector field (6.12) in this case is not additive (but it is conserved), since it consists of three components of the isospin vector of the field $(k = 1, 2, 3)$:

$$I_k = i \int \frac{\partial \mathscr{L}}{\partial \xi^r_{,t}} (I_k)^r_s \xi^s \, d^3x + \text{c.c.}, \tag{6.21}$$

where c.c. stands for a term that is the complex conjugate of the first term. In particular, for $k = 3$ we get

$$I_3 = i \int \left(\frac{\partial \mathscr{L}}{\partial \xi^0_{,t}} \xi^0 - \frac{\partial \mathscr{L}}{\partial \xi^2_{,t}} \xi^2 \right) d^3x + \text{c.c.} \tag{6.22}$$

According to (6.12), $\xi^2 = -\xi_0^*$, and therefore the first term in (6.22) is real and equal to the second term, so that instead of (6.22) we can write

$$I_3 = 2i \int \left(\frac{\partial \mathscr{L}}{\partial \xi^0_{,t}} \xi^0 - \frac{\partial \mathscr{L}}{\partial \xi^2_{,t}} \xi^2 \right) d^3x = 2i \int \frac{\partial \mathscr{L}}{\partial \xi_{,t}} \mathrm{I}_3 \xi \, d^3x. \tag{6.23}$$

Similar equalities exist for the other isospin components of the field, so that in general form

$$I_k = 2i \int \frac{\partial \mathscr{L}}{\partial \xi_{,t}} \mathrm{I}_k \xi \, d^3x. \tag{6.24}$$

Both in (6.23) and in (6.24) the integrand is the matrix product of the row $\partial \mathscr{L}/\partial \xi_{,t}$ of matrix I_k and the column ξ. Note that the total projection of the isospin vector of the field, (6.24), does not coincide with the eigenvalues (6.19), since the field is a continuous system, while (6.19) corresponds to single-particle quantum mechanical states. Only second quantization (see Chapter III) makes it possible to determine the states of a field involving a definite number, $N = 0$, 1, 2, . . ., of particles; in such states the total isospin of the field is the sum of single-particle values (6.19).

Let us now consider the spinor representation of the group of isotopic rotations, SU (2), in the space

$$c = c^1 e_1 + c^2 e_2. \tag{6.25}$$

An example of such a representation is the nucleon doublet: $c_1 = p$ (proton) and $c^2 = n$ (neutron). The corresponding generators,

$$\mathrm{X}_k = -i\sigma_k/2 \quad (k = 1, 2, 3), \tag{6.26}$$

are connected with the Pauli matrices σ_k (see Section 5), which operate in the isotopic space (6.25). The hermitian generators

$$\mathrm{I}_k = i\mathrm{X}_k = \sigma_k/2 \tag{6.27}$$

define the isospin operator in the spinor representation of the isotopic group with the same commutation relations (6.18) as in the vector representation.

Let us find the isotopic vector of the isospin field (6.25):

$$I_k = -\int \frac{\partial \mathscr{L}}{\partial c^r_{,t}} (\mathrm{X}_k)^r_s c^s \, d^3x + \text{c.c.} = \frac{i}{2} \int \frac{\partial \mathscr{L}}{\partial c^r_{,t}} (\sigma_k)^r_s c^s \, d^3x + \text{c.c.} \tag{6.28}$$

(this quantity is conserved). The fact that $\mathscr{L}$ is real and isotopically invariant implies, obviously, that

$$\frac{\partial \mathscr{L}}{\partial c^1_{,t}} c^1 = \frac{\partial \mathscr{L}}{\partial c^{*2}_{,t}} c^{*2},$$

whereby for the third component we find

$$I_3 = \frac{i}{2} \int \left(\frac{\partial \mathscr{L}}{\partial c^1_{,t}} c^1 - \frac{\partial \mathscr{L}}{\partial c^2_{,t}} c^2 \right) d^3x + \text{c.c.} = i \int \left(\frac{\partial \mathscr{L}}{\partial c^1_{,t}} c^1 - \frac{\partial \mathscr{L}}{\partial c^2_{,t}} c^2 \right) d^3x, \tag{6.29}$$

and similarly for the other components. The general result is

$$I_k = 2i \int \frac{\partial \mathscr{L}}{\partial c_{,t}} \mathrm{I}_k c d^3x, \tag{6.30}$$

where the isospin operator is

$$\mathrm{I}_k = \frac{1}{2} \sigma_k. \tag{6.31}$$

Thus, for the isospin field we arrive at an expression for the isospin of the field, (6.30), that coincides with that for the isovector field, (6.24).

The eigenvalues of the matrix $\mathrm{I}_3 = (1/2)\,\sigma_3$ are equal to $+1/2$ and $-1/2$, as is well known. They determine the single-particle quantum mechanical states of the isospin field, $c^1 = p$ and $c^2 = n$, respectively. The total isospin of the fied can be meaningfully interpreted only after second quantization of the field is carried out.

The neutron and the proton have an electric charge of 0 and $+1$, respectively. We see that these values are shifted in relation to the isospin projections $I_3 = -1/2$ and $+1/2$ by 1/2:

$$Q = I_3 + 1/2. \tag{6.32}$$

6.2 Dynamical Symmetries

Today elementary particle theory employs broader symmetries than SU (2). For instance, one generalization of SU (2) is the group of special (unimodular) unitary matrices in three-dimensional complex space, SU (3). The group space for this group is eight-dimensional. Indeed, eighteen real numbers, which parametrize the 3-by-3 general matrix of the SU (3) group, are connected by one unimodularity condition and nine unitarity conditions (nine equations for real numbers). The representation of the lowest dimension (not including the scalar representation) is realized by a triplet of complex components of the vectors of the space in which the SU (3) matrices operate.

Such groups have been related to dynamical symmetries, which manifest themselves in the interaction of fields, precisely, the weak, the electromagnetic, and the strong interaction. Here we will not attempt to provide a complete picture of the current state of the theory of structure and interactions of elementary particles, since this would lead us outside the scope of this book. Rather, we wish to demonstrate how group-theoretic methods can be applied to the classification of strongly interacting elementary particles, or hadrons (the term introduced by the Soviet physicist L.B. Okun). The known hadrons can be grouped into different multiplets corresponding to the various representations of the SU (3) group. The isotopic group SU (2) is a more simple subgroup of SU (3). Earlier we had the opportunity to see the connection between the charge of the components with the projection I_3 of the isospin, formulas (6.20) and (6.32), when we studied the multiplets of the SU (2) triplet of pions (π^-, π^0, π^+) and the doublet of nucleons (n, p). There are also other SU (2) isomultiplets of hadrons: the doublets of xi particles (Ξ^0, Ξ^-), K mesons (K^+, K^0) and their antiparticles, that is, $\overline{K}$ mesons ($\overline{K}^0$, K^-), the triplet of Σ baryons (Σ^+, Σ^0, Σ^-), and the singlets of the Λ^0 baryon and the η^0 meson.

In the SU (3) group these particles are grouped into octets, the vector representations. The baryon octet consists of the following particles: p, n, Λ^0, Σ^+, Σ^0, Σ^-, Ξ^0, and Ξ^-, while the meson octet consists of the following particles: K^+, K^0, K^-, $\overline{K}^0$, π^+, π^-, π^0, and η^0. There are also representations of higher dimensions, notably the decuplet of unstable particles, or resonances.

In the SU (3) group one is forced to introduce another additive quantum number, the hypercharge Y, which together with I_3 determines the electric charge of the components in a multiplet:

$$Q = I_3 + Y/2. \tag{6.33}$$

For pions the hypercharge Y is zero, and for nucleons it is 1, in accordance with (6.20) and (6.32). The hypercharge is the sum of two other quantum numbers, the baryon number B and the strangeness S, or $Y = B + S$. Strangeness was introduced to account for the behavior of particles in weak interactions, slow decay of "strange" hadrons into "nonstrange" particles. Heavy hadrons, or baryons, are assigned a baryon number of unity, while the relatively light hadrons, or mesons, are assigned a baryon number of zero.

Today the theory of elementary particles is based on the assumption (which indirectly has been verified by experiments) that all matter consists of elementary building blocks known as quarks: the light quarks u and d (the "up" quark and the "down" quark), the heavy quarks s and c (the "strange" quark and the "charm" quark),

and superheavy b and t quarks (the "bottom", or "beauty", quark and the "top", or "truth", quark; the last quark has not manifested itself even indirectly). The u, d, and s quarks have fractional electric charges: 2/3, —1/3, and —1/3, respectively; fractional hypercharges: 1/3, 1/3, and —2/3, respectively; and fractional baryon numbers equal to 1/3 (—1/3 for the antiquarks). From these quarks we can build the common hadrons π, K, p, n, etc., with baryons consisting of three quarks and mesons of one quark and one antiquark.

In strong interactions the $SU(2)$ and SU (3) symmetries prove to be broken, that is, inside a given multiplet the masses of particles prove to be the same only approximately. This is due to the difference in masses of the quarks: the greater the difference, the stronger the corresponding symmetry is broken. The masses of quarks are due to spontaneous symmetry breaking in weak interactions (see Section 11.3).

The c quarks possess a new quantum number C called "charm". They form hadrons with hidden charm, such as the J/ψ meson (it consists of c and $\bar{c}$), and states where the charm property is clearly evident, such as the D meson and the F meson. The b quarks form the the massive upsilon particle $\Upsilon \sim (b\bar{b})$. The introduction of this quantum number, charm, changes the basic formula (6.33) for the charges of quarks to

$$Q = I_3 + \frac{1}{2} Y + \frac{2}{3} C. \tag{6.34}$$

Each quark comes in one of the three varieties identical in all measurable qualities but different in an additional quantum number known as color. The introduction of this quantum number is necessary for Pauli's exclusion principle to be satisfied when fermions are constructed out of quarks. The color group is an SU (3) and is denoted by $SU(3)_c$. A principle that has yet to be proved theoretically but which lies at the base of the theory of hadrons is that all observable hadron states must be colorless. Thus, quarks cannot be observed in a free state; this constitutes what is known as the quark confinement hypothesis. An important property of the color group $SU(3)_c$ is that it is a local group (see Section 11). This enables building a theory of strong interactions along the same lines as the electromagnetic interaction theory by introducing an octet of vector fields, gluons, which are assumed to be the mediators in the strong interaction of elementary particles, just as photons are mediators in the electromagnetic interactions. The corresponding theory has become known as quantum chromodynamics (QCD).

7. The Scalar Field

We can now employ the methods we mastered when studying fields and construct concrete field Lagrangians for the main representations of the Lorentz group, the scalar, vector, and spinor representations. But before we do this, let us consider the general conditions that the Lagrangian density $\mathscr{L}$ must satisfy in any field theory.

7.1 Invariant Field Equations

The first requirement is that $\mathscr{L}$ be relativistically invariant, that is, invariant under transformations belonging to the Poincaré group, which includes translations and proper Lorentz transformations. Since the element of a 4-volume is relativistically invariant, the action integral S is invariant together with $\mathscr{L}$. This requirement leads to the conservation of the total 4-momentum of interacting fields.

The second requirement assumes invariance of $\mathscr{L}$ under other transformation groups, such as gauge transformations and various discrete operations, C (see Section 21), P, and T. The invariance under gauge transformations ensures that the Lagrangian and the conserved currents are real.

The third requirement assumes that for free fields the Lagrangian density $\mathscr{L}$ must be a quadratic form involving the field functions and their derivatives. In this case the field equations are linear, which ensures that the superposition principle for free fields is satisfied.

Interacting fields must be described by terms of a higher order in the Lagrangian, since every quadratic form can be reduced, via an appropriate linear transformation, to the normal form, in which the independent fields are separated.

Finally, the fourth requirement assumes that $\mathscr{L}$ depends only on those derivatives of the field functions whose order is no higher than the first. In this case the field equations are second-order differential equations.

The following (quite obvious) assertion holds true: if $\mathscr{L}$ is an invariant of the group of transformations

$$x \to x', \; u(x) \to u'(x'), \tag{7.1}$$

then the corresponding equations are invariant, too. To prove this assertion, we must first establish the meaning of the concept of invariance of an equation. Let us write the differential equation for field $u(x)$ in the following symbolic form

$$\mathrm{L}(x)\, u(x) = 0, \tag{7.2}$$

where L (x) is a differential linear operator. If after transformation (7.1) has been carried out the equation does not change its form, that is,

$$\mathrm{L}\,(x')\,u'\,(x') = 0, \tag{7.3}$$

where L (x') is the same differential operator as in (7.2) but expressed in terms of the new coordinates, then Eq. (7.2) is said to be invariant.

Now let us assume that $\mathscr{L}$ is an invariant, that is,

$$\mathscr{L}\,(x,\ u\,(x),\ \partial u/\partial x) = \mathscr{L}\,(x',\ u'\,(x'),\ \partial u'/\partial x'). \tag{7.4}$$

The Lagrange-Euler equations have the same form in the new and old coordinates:

$$\frac{\partial \mathscr{L}}{\partial u} - \frac{\partial}{\partial x^{\nu}} \frac{\partial \mathscr{L}}{\partial \partial u/\partial x^{\nu}} = 0, \tag{7.5}$$

$$\frac{\partial \mathscr{L}}{\partial u'} - \frac{\partial}{\partial x'^{\nu}} \frac{\partial \mathscr{L}}{\partial \partial u'/\partial x'^{\nu}} = 0. \tag{7.6}$$

In view of (7.4), the functional dependence of $\mathscr{L}$ on the old and new coordinates is the same and (7.5) coincides with (7.6) in form, which proves the above assertion.

Clearly, there is no general method that enables constructing Lagrangians for different fields unambiguously. Therefore, we will start from the known field equations and construct the Lagrangian for these equations. First let us take up the case of the simplest representation of the Lorentz group possible, the scalar field.

7.2 The Scalar Equation

The well-known Klein-Gordon equation

$$(\Box + m^2)\,\varphi\,(x) = 0 \tag{7.7}$$

is the linear invariant second-order differential equation for the real scalar function $\varphi\,(x)$. Here the d'Alembertian operator

$$\Box = \partial_{\mu}\partial^{\mu} = \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{7.8}$$

is, obviously, a Lorentz-invariant operator, and the scalar $m^2 > 0$ is the square of the mass of the particle associated with field $\varphi\,(x)$. If we introduce the 4-momentum operators $\mathrm{p}_{\mu} = i\partial/\partial x^{\mu}$, we can rewrite (7.7) as follows:

$$(\mathrm{p}^2 - m^2)\,\varphi\,(x) = 0, \quad \text{where } \mathrm{p}^2 = \mathrm{p}_{\mu}\mathrm{p}^{\mu}. \tag{7.9}$$

The Lagrangian density of the field $\varphi\,(x)$ can be chosen in the form

$$\mathscr{L} = \frac{1}{2} \frac{\partial \varphi}{\partial x^{\mu}} \frac{\partial \varphi}{\partial x_{\mu}} - \frac{m^2}{2} \varphi^2\,(x), \tag{7.10}$$

which yields Eq. (7.7).

Let us find the density of the canonical momentum

$$\pi = \frac{\partial \mathscr{L}}{\partial \varphi_{,t}} = \frac{\partial \varphi}{\partial t} \tag{7.11}$$

and the Hamiltonian density (the field-energy density)

$$\mathscr{H} = \pi \frac{\partial \varphi}{\partial t} - \mathscr{L} = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\varphi^2 + \frac{1}{2}(\nabla\varphi)^2. \tag{7.12}$$

Since $\mathscr{H} \geqslant 0$, we arrive at the important conclusion that the energy of a scalar field is positive definite:

$$E = \int d^3x \mathscr{H} \geqslant 0. \tag{7.13}$$

The energy-momentum tensor, according to the general formulas of Section 4, is

$$T_\mu^\nu = \frac{\partial \varphi}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial \varphi_{,\nu}} - \delta_\mu^\nu \mathscr{L} = \varphi_{,\mu}\varphi^{,\nu} - \delta_\mu^\nu \mathscr{L}. \tag{7.14}$$

The covariant tensor $T_{\mu\nu}$ is symmetric, which follows from the fact that the field is scalar. The spin of the field is clearly zero, and the orbital angular momentum

$$L_{\alpha\beta}^\nu = T_\alpha^\nu x_\beta - T_\beta^\nu x_\alpha \tag{7.15}$$

is conserved:

$$\frac{\partial L_{\alpha\beta}^\nu}{\partial x^\nu} = 0. \tag{7.16}$$

The total 4-momentum of the field has the form

$$P^\alpha = \int T^{\alpha 0}\, d^3x = (P^0,\ \mathbf{P}), \quad P^0 = \int \mathscr{H}\, d^3x, \quad \mathbf{P} = \int \varphi_{,t}\nabla\varphi\, d^3x, \tag{7.17}$$

and since the field is real, $\varphi^* = \varphi$, we conclude that the electromagnetic current is zero, $j^\mu = 0$.

For a complex scalar field $\varphi = \varphi_1 + i\varphi_2$, we must obtain independent equations for its real and imaginary parts. Instead of this the complex conjugate variables φ and φ^* can be taken as the independent components. The invariant real Lagrangian in this case is

$$\mathscr{L} = \frac{\partial \varphi^*}{\partial x^\mu} \frac{\partial \varphi}{\partial x_\mu} - m^2\varphi^*(x)\,\varphi(x), \tag{7.18}$$

while the field equations are

$$(\Box + m^2)\,\varphi(x) = 0, \quad (\Box + m^2)\varphi^*(x) = 0. \tag{7.19}$$

The canonical momenta are

$$\pi = \frac{\partial \mathscr{L}}{\partial \varphi_{,t}} = \varphi^*_{,t}, \quad \pi^* = \varphi_{,t}, \tag{7.20}$$

and the Hamiltonian density

$$\mathscr{H} = \varphi_{,t}\pi + \varphi^*_{,t}\pi^* - \mathscr{L} = \pi^*\pi + (\nabla\varphi^*)\cdot(\nabla\varphi) + m^2\varphi^*\varphi \quad (7.21)$$

is nonnegative, with the result that the energy of the field is positive definite.

The energy-momentum tensor is symmetric:

$$T_{\mu\nu} = \frac{\partial\varphi}{\partial x^\nu}\frac{\partial\mathscr{L}}{\partial\varphi_{,\mu}} + \frac{\partial\varphi^*}{\partial x^\nu}\frac{\partial\mathscr{L}}{\partial\varphi^*_{,\mu}} - g_{\mu\nu}\mathscr{L} = \varphi^*_{,\mu}\varphi_{,\nu} + \varphi_{,\mu}\varphi^*_{,\nu} - g_{\mu\nu}\mathscr{L}. \quad (7.22)$$

The components of the 4-momentum are

$$\mathbf{P} = \int d^3x\,(\varphi^*_{,t}\nabla\varphi + \varphi_{,t}\nabla\varphi^*), \quad P_0 = \int d^3x\,\mathscr{H} \geqslant 0. \quad (7.23)$$

The electromagnetic-current 4-vector is nonzero:

$$j^\mu = -i\left(\frac{\partial\mathscr{L}}{\partial\varphi_{,\mu}}\varphi - \frac{\partial\mathscr{L}}{\partial\varphi^*_{,\mu}}\varphi^*\right) = -i\,(\varphi\varphi^{*,\mu} - \varphi^*\varphi^{,\mu}). \quad (7.24)$$

The total charge of the field does not have a definite sign:

$$Q = \int d^3x j^0 = i\int\left(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}\right)d^3x. \quad (7.25)$$

As noted earlier, a meaningful interpretation of this quantity is possible only within the framework of the second quantization concept. The spin of the field, obviously, is zero. Finally, the angular momentum of the field can be represented by the same formulas as in the case of a real field (see (7.15) and (7.16)).

Let us consider the example of the charged scalar field of pions comprising the isotriplet $\xi = (\pi^-, \pi^0, \pi^+)$ (see Section 6). The isotopically invariant Lagrangian in this case is

$$\mathscr{L} = \frac{1}{2}\left(\frac{\partial\xi^*}{\partial x^\mu}\frac{\partial\xi}{\partial x_\mu} - m^2\xi^*\xi\right), \quad (7.26)$$

where

$$\xi^*\xi = |\,\xi^0\,|^2 + |\,\xi^1\,|^2 + |\,\xi^2\,|^2.$$

Putting $\varphi = \xi^2$ and taking into account the fact that $\xi^2 = -\xi^{*0}$, we obtain

$$\mathscr{L} = \partial_\mu\varphi^*\partial^\mu\varphi - m^2\varphi^*\varphi + \frac{1}{2}(\partial_\mu\xi^{*1}\,\partial^\mu\xi^1 - m^2\xi^{*1}\xi^1). \quad (7.27)$$

According to (7.24), the charge of the pion field is

$$Q = i\int d^3x\left(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}\right)d^3x. \quad (7.28)$$

By definition (6.23), the isospin of the field is

$$I_3 = i \int d^3x \, (\xi^0 \xi^{*0}_{,t} - \xi^2 \xi^{*2}_{,t}). \tag{7.29}$$

This expression coincides with (7.28) if we put $\varphi = \xi^2 = -\xi^{*0}$. Thus, the charge of the pion field is determined by the value of the projection I_3 of the isospin of the field.

Let us consider one more example that involves a self-acting charged scalar field.

In Section 1 we considered the example of a theory with a Lagrangian density of a real field u invariant under the discrete operation of reflection, $u \to -u$. We saw that in the special case of a potential energy function of the form

$$V(u) = \frac{1}{4} \lambda \left(\frac{m^2}{\lambda} - u^2 \right)^2, \quad \lambda > 0, \tag{7.30}$$

spontaneous symmetry breaking occurs (i.e. parity is not conserved), that is, the solution ceases to be invariant near one of the minima of $u = u_+$ while the Lagrangian density $\mathscr{L}$ remains invariant. The solution near $u = 0$ is unstable, and we arrive at a situation in which the solution splits into two stable solutions that do not match. The more general case occurs if we consider the case of continuous gauge symmetry.

Precisely, let us take the Lagrangian density of a complex nonlinear scalar field,

$$\mathscr{L} = \frac{\partial \varphi^*}{\partial x^\mu} \frac{\partial \varphi}{\partial x_\mu} - V(\varphi^*, \varphi) = \frac{1}{2} (\partial \varphi^1)^2 + \frac{1}{2} (\partial \varphi^2)^2 - V(\varphi^1, \varphi^2), \tag{7.31}$$

where

$$\varphi = \frac{1}{\sqrt{2}} (\varphi_1 + i\varphi_2),$$

with φ_1 and φ_2 real functions. If V depends only on the modulus of φ, that is, on

$$\rho = (\varphi^1)^2 + (\varphi^2)^2 = 2\varphi^* \varphi, \tag{7.32}$$

the Lagrangian density $\mathscr{L}$ is invariant under the gauge transformation

$$\varphi(x) \to \varphi(x) e^{i\alpha}, \quad \varphi^*(x) \to \varphi^*(x) e^{-i\alpha}, \tag{7.33}$$

which means conservation of the electric charge of the field.

Suppose that

$$V(\rho) = \frac{\lambda}{4} \left(\frac{m^2}{\lambda} - \rho^2 \right)^2. \tag{7.34}$$

We transfer to new functions ρ and ϑ in the following manner:

$$\varphi(x) = \frac{1}{\sqrt{2}} \rho(x) e^{i\vartheta(x)}, \tag{7.35}$$

where ρ and ϑ are real. Then

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \rho)^2 + \frac{1}{2}\rho^2 (\partial_\mu \vartheta)^2 - V(\rho). \tag{7.36}$$

Suppose that $m^2 > 0$ and $\lambda > 0$. Then $V(\rho)$ has a minimum at $\rho = \eta = (m^2/\lambda)^{1/2} \neq 0$ far from point 0 (if m^2 is negative, the minimum occurs at zero). For the function φ we have a set of values for arbitrary real α,

$$\varphi = \frac{1}{\sqrt{2}} \eta e^{i\alpha}, \tag{7.37}$$

for which $V(\rho)$ is minimal. Thus, the ground state of the system (vacuum state) in which $V(\rho)$ is minimal is infinitely degenerate. All states with different α's are physically equivalent, but transitions between these states are impossible, hence spontaneous symmetry breaking. We must select a definite value of α, say, $\alpha = 0$.

Hence, let us select the vacuum in the form

$$\varphi = \frac{1}{\sqrt{2}} \eta \quad (\alpha = 0). \tag{7.38}$$

Then in the vicinity of this state we have

$$\varphi(x) = \frac{1}{\sqrt{2}} (\rho'(x) + \eta) e^{i\vartheta(x)}, \tag{7.39}$$

where

$$\rho' = \rho - \eta, \quad |\rho'| \ll \eta, \quad |\vartheta(x)| \ll 1.$$

Let us expand $\mathscr{L}$ to within terms proportional to ρ'^2 and ϑ^2. The result is

$$\mathscr{L} = \text{constant} + \frac{1}{2}(\partial_\mu \rho')^2 + \frac{1}{2}\eta^2 (\partial_\mu \vartheta)^2 - \frac{\rho'^2}{2} m_\rho^2, \tag{7.40}$$

where $m_\rho = \sqrt{2}m$. The field equations are

$$(\Box + m_\rho^2)\rho' = 0, \quad \Box \vartheta = 0. \tag{7.41}$$

Thus, we have obtained two free neutral scalar fields: the first is associated with particles of mass m_ρ, and the second with massless particles. The discarded terms of a higher order must describe the interaction between these particles.

Equations (7.41) are valid for an approximate solution near the vacuum state, that is, for small amplitudes of the fields, ρ' and ϑ. If the system is excited (say, by raising its temperature), the symmet-

ry is restored due to the interaction, and the solution proves to be symmetric in phase. This mechanism of formation of massless particles under spontaneous breaking of continuous symmetry was predicted by J. Goldstone in 1961. The same mechanism manifests itself in quantum field theory, where the massless 0-spin particles that emerge in the theory with broken symmetry are known as Goldstone bosons. In Section 11 we will see how this penomenon can be generalized to include the case of local gauge symmetries; there it leads to the so-called Higgs mechanism of formation of masses of elementary particles.

8. Spinor Representation of the Lorentz Group

In this section we will first discuss the spinor representation of the proper Lorentz group, a representation that is a generalization of the spinor representation of the three-dimensional rotation group. The introduction of space inversion into the picture will enable us to write the appropriate invariant equation, the Dirac equation for the spinor field, and to study its properties.

8.1 The $SL(2, C)$ Group

The elements of the Lorentz group depend on six parameters, whereby to represent the group by 2-by-2 matrices we must drop the unitarity condition. This leaves us with the unimodularity condition. As a result we get the matrices

$$\Lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det \Lambda = 1. \tag{8.1}$$

The number of independent matrix elements is equal to $2 \times 4 - 2 = 6$, since the unimodularity condition provides two equations for the real and imaginary parts of the determinant. The special (unimodular) matrices carry out the linear transformations in two-dimensional complex space

$$a = a^1 e_1 + a^2 e_2, \quad a' = \Lambda a, \quad e' = e\Lambda \tag{8.2}$$

and constitute the SL (2, C) group.

Since the unitarity condition is dropped, that is, $\Lambda^+ \neq \Lambda^{-1}$, it is advisable to consider the second vector space

$$\dot{a} = a^{\dot{1}} e_{\dot{1}} + a^{\dot{2}} e_{\dot{2}} = e_{\dot{s}} a^{\dot{s}}, \tag{8.3}$$

in which the transformation Λ^* operates:

$$\dot{a}' = \Lambda^* \dot{a}, \quad \Lambda^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}. \tag{8.4}$$

Generally speaking, the vectors a and $\dot{a}$ may not be the complex conjugates of each other, although it could happen that $\dot{a} = a^*$. We use the base vectors e_s and $e_{s'}$ to construct the new four-dimensional space

$$\mathrm{c} = c^{11'} e_1 e_{1'} + c^{12'} e_1 e_{2'} + c^{21'} e_2 e_{1'} + c^{22'} e_2 e_{2'}. \tag{8.5}$$

The vector in such a space can be represented in the form of a matrix,

$$\mathrm{c} = \| c^{rs'} \| = \begin{pmatrix} c^{11'} & c^{12'} \\ c^{21'} & c^{22'} \end{pmatrix}. \tag{8.6}$$

In the space of vectors c the transformations Λ and Λ^* have the representation $\mathrm{c}' = \mathrm{Dc}$:

$$\begin{aligned} e_r e_{s'} &\to e'_r e'_{s'} = e_p e_{q'} \Lambda^p_r \Lambda^{*q'}_{s'}, \\ c^{rs'} &\to c'^{rs'} = \Lambda^r_p \Lambda^{*s'}_{q'} c^{pq'}. \end{aligned} \tag{8.7}$$

Since $\det \Lambda = 1$, we can write

$$\det \mathrm{c}' = \det \Lambda \det \mathrm{c} \det \Lambda^* = \det \mathrm{c},$$

which yields the invariant of such transformations:

$$c^{11'} c^{22'} - c^{21'} c^{12'} = \mathrm{inv}. \tag{8.8}$$

Let us consider the case where the space $\dot{a}$ is the complex conjugate of space a. Then

$$\mathrm{c} = c^{11'} |e_1|^2 + c^{12'} e_1 e_2^* + c^{21'} e_2 e_1^* + c^{22'} |e_2|^2. \tag{8.9}$$

If we require that

$$c^{11'} = c^{*11'}, \quad c^{22'} = c^{*22'}, \quad c^{12'} = c^{*21'}, \tag{8.10}$$

then $\mathrm{c}^* = \mathrm{c}$. Next we introduce the real quantities

$$\begin{aligned} x &= \frac{1}{2}(c^{12'} + c^{21'}), \quad y = \frac{i}{2}(c^{12'} - c^{21'}), \\ z &= \frac{1}{2}(c^{11'} - c^{22'}), \quad t = \frac{1}{2}(c^{11'} + c^{22'}). \end{aligned} \tag{8.11}$$

They are the components of a vector of the space of equivalent representation, which is related to the initial one by a similarity transformation:

$$\mathrm{V} \begin{pmatrix} c^{11'} \\ c^{12'} \\ c^{21'} \\ c^{22'} \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, \quad \mathrm{D}' = \mathrm{VDV}^{-1}, \tag{8.12}$$

where the matrix of the transformation has the form

$$V=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{i}{2} & -\frac{i}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}. \tag{8.13}$$

The conditions (8.10) and the reality of x, y, z, and t are invariant under transformations belonging to the SL (2, C) group. Indeed, if $c = c^*$, the vector c can be represented in the form of the product of two vectors:

$$c = (a^1 e_1 + a^2 e_2)\,(a^{*1} e_1^* + a^{*2} e_2^*),$$

from which it follows that x, y, z, and t are real under all transformations belonging to this group.

The invariant of this group in the new coordinates can be written thus:

$$c^{11'}c^{22'} - c^{12'}c^{21'} = t^2 - x^2 - y^2 - z^2 = \text{inv}. \tag{8.14}$$

From this it follows that the matrices $D' = VDV^{-1}$ are real matrices of the proper Lorentz group $L_+^\uparrow$. We will now show that the D representation contains all the proper transformations of the Lorentz group.

(1) If we restrict our discussion to transformations with the unitary matrix $\Lambda = U$, for which $\alpha^* = \delta$ and $\beta^* = -\gamma$, then

$$t = \frac{1}{2}(c^{11'} + c^{22'}) = \text{inv}, \tag{8.15}$$

since

$$c^{11'} + c^{22'} \sim c^1 c^{*1} + c^2 c^{*2} = \text{inv}.$$

For the unitary group,

$$e_1' = e_1^* \sim e_1, \quad e_2' = e_2^* \sim -e_1. \tag{8.16}$$

Whence

$$\begin{aligned} c &\sim -c^{12'}e_1^2 + (c^{11'} - c^{22'})\,e_1 e_2 + c^{21'}e_2^2 \\ &= -(x - iy)\,e_1^2 + 2z e_1 e_2 + (x + iy)\,e_2^2, \end{aligned}$$

that is, c is transformed just like in the case of transformations belonging to O (3), and (as shown in Section 5) these transformations exhaust the O (3) group.

(2) Let us now select Λ in the following form:

$$\Lambda = \Lambda_{\tilde{\varphi}^3} = \begin{pmatrix} e^{\tilde{\varphi}^3/2} & 0 \\ 0 & e^{-\tilde{\varphi}^3/2} \end{pmatrix}, \tag{8.17}$$

where $\tilde{\varphi}^3$ is a real parameter. This matrix satisfies the condition $\det \Lambda = 1$, but it is not unitary: $\Lambda^* \neq \Lambda^{-1}$. The vector c is transformed according to (8.7). Let us spread out matrix (8.6) into a column whose elements are $c^{11'}$, $c^{12'}$, $c^{21'}$, and $c^{22'}$. For the matrix D, which carries out the transformation $c' = Dc$, (8.7), we get

$$D = \begin{pmatrix} e^{\tilde{\varphi}^3} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-\tilde{\varphi}^3} \end{pmatrix}. \tag{8.18}$$

Going over to the equivalent representation (8.12), we find that

$$D' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \tilde{\varphi}^3 & \sinh \tilde{\varphi}^3 \\ 0 & 0 & \sinh \tilde{\varphi}^3 & \cosh \tilde{\varphi}^3 \end{pmatrix}. \tag{8.19}$$

This is precisely the matrix of pseudorotations in the 03-plane through an angle $\tilde{\varphi}^3$, which corresponds to motion with velocity $v/c = \tanh \tilde{\varphi}^3$.

In the spinor space, the matrix Λ specified in (8.17) has corresponding to it the generator

$$\tilde{X}_3 = \left(\frac{\partial \Lambda}{\partial \tilde{\varphi}^3} \right)_{\tilde{\varphi}^3 = 0} = \frac{1}{2} \sigma_3. \tag{8.20}$$

As shown in Section 3, one can use the subgroup of pseudorotations in the 03-plane together with the $O(3)$ group of real rotations to obtain all the transformations belonging to the proper Lorentz group. Thus, we have obtained a representation of the $SL(2, C)$ group by matrices belonging to the $L_+^\uparrow$ group.

The inverse representation $L_+^\uparrow \to SL(2, C)$, that is, the spinor representation of the proper Lorentz group, is two-valued, just as the spinor representation of the $O(3)$ group is. Here the orthogonal rotations, as noted in Section 5, are represented by the matrices

$$\Lambda_\varphi = \exp\left(-\frac{i}{2} \varphi^k \sigma_k \right), \tag{8.21}$$

while pseudorotations are represented by the matrices

$$\Lambda_{\underset{\sim}{\varphi}} = \exp\left(\frac{1}{2}\tilde{\varphi}^k \sigma_k\right), \tag{8.22}$$

which are obtained from (8.17) via a certain real rotation (see Section 3):

$$\Lambda_{\underset{\sim}{\varphi}} = \Lambda'_{\varphi} \Lambda_{\underset{\sim}{\varphi}^3} \Lambda_{\varphi}. \tag{8.23}$$

The corresponding generators have the form

$$X_k = -\frac{i}{2}\sigma_k, \quad \tilde{X}_k = \frac{1}{2}\sigma_k. \tag{8.24}$$

The reader can easily see by a direct check that the commutation relations for the generators (8.24) are the same as for the 4-by-4 matrices introduced in Section 3 (see (3.86)-(3.88)), as was to be expected.

Note that according to (8.6) and (8.11) a 4-vector in Minkowski's space-time (t, x, y, z) has corresponding to it a spinor

$$\| c^{rs'} \| = \begin{pmatrix} z+t & x-iy \\ x+iy & -z+t \end{pmatrix}, \tag{8.25}$$

or, if we employ the Pauli matrices,

$$c = \sigma_\mu x^\mu = \sigma_0 t + \sigma_k x^k, \tag{8.26}$$

where $\sigma_0 = I$ is the unit matrix. The reader will recall that a similar representation was obtained when we restricted the space of vectors c by conditions (8.10) combined with the requirement that the initial two-dimensional spaces be complex conjugate: $\dot{a} = a^*$. Let us now drop this requirement and introduce two different spaces with base vectors

$$e_1, e_2 \text{ and } e_{1'}, e_{2'} \neq e_1^*, e_2^*. \tag{8.27}$$

We now introduce an extended space that includes the two spaces we have just mentioned. We denote the new base vectors thus:

$$e_{1'} = g^{2'}, \quad e_{2'} = -g^{1'}. \tag{8.28}$$

Then a vector in such an extended space can be written in the form

$$a + \dot{a} = a^r e_r + a_{s'} g^{s'}. \tag{8.29}$$

Such an extension of the space will make it possible to complement the spinor representation of group $L_+^\uparrow$ to the representation of the improper Lorentz group, which also includes space inversion P (see Section 2).

Let us first see how inversion operates in the vector representation of the $L_+^\uparrow$ group. We construct the operator $PAP^{-1} = PAP$, where A is the Lorentz matrix, and for vectors, obviously, $PP = 1$ and

$P^{-1} = P$. Then the matrix of operator PAP^{-1} will differ, as can easily be demonstrated, from the Lorentz matrix by the signs of the first row and the first column, with the block in the matrix responsible for spatial orthogonal rotations remaining unchanged. This means that the inversion transformation is commutative with all spatial rotations but is not commutative with space-time rotations, since inversion transforms the special Lorentz transformation into a transformation with the sign of the velocity reversed, $\beta \to -\beta$.

In view of the fact that the spinor representation is two-valued, the operator of spinor inversion, P, may be chosen in two ways: either in such a manner that applying the operation of inversion twice does not change the spinor $P^2 = 1$, or in such a manner that the sign of the spinor is reversed, $P^2 = -1$. Taking up the second possibility, we define the inversion operation as follows:

$$Pa_r = ia^r, \quad Pa^s = ia_{s'} \tag{8.30}$$

and, respectively,

$$Pe_r = ig^{r'}, \quad Pg^{s'} = ie_s. \tag{8.31}$$

Such a definition of P ensures the commutativity of inversion with all spatial rotations U, since $a_{r'}$ and a^r are transformed in the same way under U (see (8.16) and (8.28)). At the same time the reader can easily see that the operation (8.30) reverses the sign of r. Indeed, recalling the representation of a 4-vector in the form (8.25), $c = \sigma_\mu x^\mu$, and allowing for the fact that under a P operation

$$c^{11'} \sim a^1 a_{2'} \to -a_{1'} a^2, \quad c^{22'} \sim -a^2 a_{1'} \to a_{2'} a^1,$$

we get

$$\frac{1}{2}(c^{11'} - c^{22'}) = z \to -z.$$

Similarly, for the other components we have $x \to -x$, $y \to -y$. This means that the operation (8.30) is inversion.

Now let us build another matrix, c′:

$$\| c'_{s'r} \| = \begin{pmatrix} -z+t & -x+iy \\ -x-iy & z+t \end{pmatrix}. \tag{8.32}$$

Here $c'c = \det c \times I$, where I is the unit matrix, and $\det c = t^2 - r^2 = x^\mu x_\mu$. We can write the expression for the c′ matrix using the Pauli matrices:

$$c' = \sigma'_\mu x^\mu, \quad \sigma' = (I, -\sigma_k), \tag{8.33}$$

Due to inversion, c transforms into c′ and vice versa, c′ into c. Multiplying matrix c by matrix c′ yields

$$cc' = \sigma_\mu x^\mu \sigma'_\nu x^\nu = \frac{1}{2}(\sigma_\mu \sigma'_\nu + \sigma'_\nu \sigma_\mu) x^\mu x^\nu,$$

where we symmetrized the product of σ and σ' matrices. Allowing for the fact that $cc' = x_\mu x^\mu \equiv g_{\mu\nu}x^\mu x^\nu$, we find that

$$\sigma_\mu\sigma'_\nu + \sigma'_\nu\sigma_\mu = 2Ig_{\mu\nu}. \tag{8.34}$$

Thus, the introduction of inversion into the Lorentz group requires extending the representation space (8.29). This extended space consists of so-called bispinors, that is, four-component spinors with elements a_1, a_2, $a^{1'}$, $a^{2'}$. The space is invariant under transformations P and $L_+^\uparrow$, and each of the subspaces a and $\dot{a}$ is invariant under $L_+^\uparrow$. Note that the rules (8.30) and (8.31) correspond to P^2 being equal to $-I$. Similar rules for the inversion of spinors can be built with $P^2 = I$.

8.2 The Dirac Equation

Let us construct combinations that are covariant with respect to the SL (2, C) group of the form

$$a^r = c^{rs'}b_{s'}, \tag{8.35}$$

$$a_{s'} = c'_{s'r}b^r. \tag{8.36}$$

This system of equations is invariant under a combination of transformations belonging to SL (2, C) and P. Indeed, under P Eq. (8.35) transforms into (8.36) and vice versa, while each of these equations is invariant under SL (2, C), since a, b, and c are covariant quantities.

Let us select matrix c and the bispinors a and b in the form

$$c = \sigma_\mu\partial^\mu, \quad b = \frac{i}{m}a, \quad \dot{b} = \frac{i}{m}\dot{a}, \tag{8.37}$$

where $\partial^\mu = (\partial^0, -\nabla)$ is the differential operator. Then instead of (8.35) and (8.36) we get

$$ma = i\sigma_\mu\partial^\mu\dot{a}, \quad m\dot{a} = i\sigma'_\mu\partial^\mu a. \tag{8.38}$$

We denote a bispinor by $\psi = \psi(x)$ and introduce the 4-by-4 γ-matrices thus:

$$\psi = \begin{pmatrix} a \\ \dot{a} \end{pmatrix}, \quad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma'_\mu & 0 \end{pmatrix}. \tag{8.39}$$

Then the system of equations (8.38) can be written in compact form:

$$(i\gamma_\mu\partial^\mu - m)\psi = 0. \tag{8.40}$$

This spinor equation, invariant under transformations belonging to the Lorentz group, was suggested by P.A.M. Dirac in 1928 and became known as the Dirac equation.

Recalling the property (8.34) of σ_μ and σ'_μ, we can rewrite (8.38) as a second-order equation,

$$(\Box + m^2)\psi = 0, \tag{8.41}$$

from which it follows that m is the mass of the particle described by the spinor equation. Obviously, (8.34) leads to the following anticommutation relation for the γ-matrices:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}. \tag{8.42}$$

Equation (8.41) can be obtained also from (8.40) by applying operator $i\gamma_\mu\partial^\mu + m$ to the latter and employing (8.42).

The explicit form of the γ-matrices follows from the definitions $\sigma_\mu = (\mathrm{I}, \sigma_k)$ and $\sigma'_\mu = (\mathrm{I}, -\sigma_k)$:

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad \gamma^k = -\gamma_k, \ \gamma^0 = \gamma_0. \tag{8.43}$$

Let us consider the hermitian-conjugate equation

$$\psi^+(-i\gamma_\mu^+\partial^\mu - m) = 0, \tag{8.44}$$

where ∂^μ acts from right to left. We introduce the Dirac-conjugate quantities

$$\bar{\gamma}_\mu = \gamma^0\gamma_\mu^+\gamma^0, \quad \bar{\psi} = \psi^+\gamma^0. \tag{8.45}$$

Note that

$$\bar{\gamma}_\mu = \gamma_\mu, \text{ since } \gamma_k^+ = -\gamma_k, \quad \gamma_0^+ = \gamma_0. \tag{8.46}$$

Then instead of (8.44) we get the Dirac-conjugate equation

$$\bar{\psi}(i\gamma_\mu\partial^\mu + m) = 0. \tag{8.47}$$

Note that the matrices γ^μ and the bispinor ψ are defined to within a unitary transformation, that is, a similarity transformation

$$\gamma_\mu \to \mathrm{V}\gamma_\mu\mathrm{V}^+, \quad \psi \to \mathrm{V}\psi, \quad \mathrm{V}^+\mathrm{V} = \mathrm{I}.$$

The representation we have just obtained is known as the spinor representation. To study the motion of Dirac (spinor) particles another representation is usually employed, the standard representation, which is obtained from the spinor representation by applying the operator

$$\mathrm{V} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{I} & -\mathrm{I} \end{pmatrix}.$$

In this representation,

$$\gamma^0 = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \gamma^k = -\gamma_k, \tag{8.48}$$

while the bispinor has the form

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a + \dot{a} \\ a - \dot{a} \end{pmatrix}, \tag{8.48a}$$

with the spinors φ and χ belonging to subspaces invariant under three-dimensional rotations and inversion:

$$\begin{aligned} U\varphi = \varphi' &= \frac{1}{\sqrt{2}}(a' + \dot{a}'), \\ U\chi = \chi' &= \frac{1}{\sqrt{2}}(a' - \dot{a}'), \\ P\varphi = i\varphi, \quad P\chi &= -i\chi. \end{aligned} \tag{8.49}$$

At the same time, the space-time rotations $\Lambda_{\varphi}^{\sim}$ lead to a mixing of φ and χ, since a and $\dot{a}$ are transformed differently under such rotations:

$$\begin{aligned} a'^{r} &= \Lambda_{p}^{r} a^{p}, \\ \dot{a}'_{s'} &= (\Lambda^{*-1})_{s'}^{q'} \dot{a}_{q'}. \end{aligned} \tag{8.50}$$

The first rule is the initial definition of SL (2, C) transformations, while the second is obtained if we recall the definition of spinor $\dot{a}$ (8.28), (8.29). For unitary matrices corresponding to rotations, $U^{+} = U^{-1}$, and the second rule yields the same result as the first.

Recalling the explicit form of the matrices of the Lorentz transformations (8.21) and (8.22) for the bispinor (8.48a), we arrive at the following transformation law:

$$\begin{aligned} \psi' &= \Lambda\psi, \quad \Lambda = (\Lambda_{\varphi}, \Lambda_{\tilde{\varphi}}), \\ \Lambda_{\varphi} &= \exp\left(-\frac{i}{2}\varphi^{k}\Sigma_{k}\right), \quad \Lambda_{\tilde{\varphi}} = \exp\left(\frac{1}{2}\tilde{\varphi}^{k}\alpha_{k}\right), \end{aligned} \tag{8.51}$$

where the matrices Σ_k and α_k in the standard representation have the form

$$\Sigma_{k} = \begin{pmatrix} \sigma_{k} & 0 \\ 0 & \sigma_{k} \end{pmatrix}, \quad \alpha_{k} = \begin{pmatrix} 0 & \sigma_{k} \\ \sigma_{k} & 0 \end{pmatrix}. \tag{8.52}$$

The inversion operator in the standard representation is

$$P = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} = i\gamma^{0}. \tag{8.53}$$

Note that besides accounting for the result of the operation of matrix (8.53) on the bispinor, which depends on coordinates $\mathbf{x}$, we

must also replace x with —x in the bispinor. Let us introduce the matrices

$$\sigma_{\alpha\beta} = \frac{i}{2}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha) \tag{8.54}$$

($\alpha, \beta = 0, 1, 2, 3$). Then

$$\sigma_{0i} = i\gamma_0\gamma_i = -i\alpha_i \ (\gamma^0\gamma^i = \alpha_i), \tag{8.55}$$

$$\sigma_{ik} = i\gamma_i\gamma_k = \varepsilon_{ikj}\Sigma_j, \tag{8.56}$$

and instead of matrices Λ_φ and $\Lambda_{\tilde{\varphi}}$ (8.51) we introduce a more compact way of writing Lorentz transformations in the spinor representation:

$$\Lambda = \exp\left(-\frac{i}{2}\sigma_{\alpha\beta}\varphi^{\alpha\beta}\right), \tag{8.57}$$

where the parameters $\varphi^{\alpha\beta}$ are defined in the same manner as in Section 3: $\varphi^{12} = \varphi^3$, etc., and $\varphi^{01} = \tilde{\varphi}^1$, etc. Accordingly, we obtain six generators of the Lorentz group,

$$X_{\alpha\beta} = -\frac{i}{2}\sigma_{\alpha\beta}, \tag{8.58}$$

in the spinor representation.

9. The Spinor Field and the Massive Vector Field

The scalar field considered in Section 7 was one-component. This is the simplest possible example of a field that realizes a representation of the Lorentz group, the scalar representation. More complicated examples include the spinor, or Dirac, field and the vector field, which yield the spinor and vector representations of the Lorentz group.

Examples of spinor fields are the electron-positron field, the neutrino field, the muon field. As a rule, vector fields correspond to particles that are the carriers of interactions, that is, mesons. Such are the massive vector bosons W^+, W^-, Z^0 of the weak interaction, and photons, the massless quanta of electromagnetic fields. It must be noted, however, that a massless vector field requires a special investigation, which will be carried out in Section 10.

9.1. The Lagrangian Density and Noether's Currents of the Dirac Field

The invariant Dirac equations*

$$(i\gamma\partial - m)\psi = 0, \quad \bar{\psi}(i\gamma\partial + m) = 0 \tag{9.1}$$

* Here and in what follows we use the compact notation $\gamma\partial \equiv \gamma_\mu \frac{\partial}{\partial x_\mu}$.

can be obtained from the Lagrangian density

$$\mathscr{L} = \frac{i}{2}(\bar{\psi}\gamma_\mu \partial^\mu \psi - (\partial_\mu \bar{\psi})\gamma^\mu \psi) - m\bar{\psi}\psi$$

$$= \frac{i}{2}(\bar{\psi}\gamma\,\partial\psi - (\partial\bar{\psi})\gamma\psi) - m\bar{\psi}\psi. \tag{9.2}$$

We note that in view of the field equations (9.1), the Lagrangian density of the Dirac field, (9.2), vanishes, $\mathscr{L} = 0$. This means that in the definition of the canonical energy-momentum tensor we can drop the last diagonal term proportional to $\mathscr{L}$. The result is

$$T_\alpha^\nu = \frac{\partial \mathscr{L}}{\partial \psi_{,\nu}} \psi_{,\alpha} + \bar{\psi}_{,\alpha} \frac{\partial \mathscr{L}}{\partial \bar{\psi}_{,\nu}}.$$

Substituting the values of the derivatives, which follow from the definition (9.2) of the Lagrangian density, we get

$$T_\alpha^\nu = \frac{i}{2}(\bar{\psi}\gamma^\nu \psi_{,\alpha} - \bar{\psi}_{,\alpha}\gamma^\nu \psi). \tag{9.3}$$

The $T_{\alpha\nu}$ tensor is not symmetric since the spin of the field differs from zero. Let us find the 4-momentum of the field,

$$P^\alpha = \int T^{\alpha 0} d^3x = \frac{i}{2}\int (\bar{\psi}\gamma^c \psi^{,\alpha} - \bar{\psi}^{,\alpha}\gamma^0\psi)\, d^3x$$

$$= \frac{i}{2}\int (\psi^+ \psi^{,\alpha} - \psi^{+,\alpha}\psi)\, d^3x. \tag{9.4}$$

Transforming the second term, we can write

$$P^\alpha = \frac{i}{2}\int d^3x \left(\psi^+ \psi^{,\alpha} - \frac{\partial}{\partial x_\alpha}(\psi^+\psi) + \psi^+ \psi^{,\alpha}\right)$$

$$= i\int d^3x \psi^+ \psi^{,\alpha} + \frac{i}{2}\int d^3x \frac{\partial}{\partial x_\alpha}(\psi^+\psi). \tag{9.5}$$

The second term on the right-hand side is zero. Indeed, at $\alpha = 1, 2, 3$ it transforms into an integral over the surface that limits the volume occupied by the field and, due to the boundary conditions for the field, vanishes. At $\alpha = 0$ the derivative $\partial/\partial t$ can be taken outside the integral sign, and we get

$$\frac{\partial}{\partial t}\int \psi^+\psi\, d^3x. \tag{9.6}$$

The integrand determines the density of the field charge. This follows from the definition of the density of the 4-current,

$$j^\mu = -i\left(\frac{\partial \mathscr{L}}{\partial \psi_{,\mu}}\psi - \bar{\psi}\frac{\partial \mathscr{L}}{\partial \bar{\psi}_{,\mu}}\right),$$

which in the case of a Dirac field is

$$j^\mu = \bar{\psi}\gamma^\mu\psi. \tag{9.7}$$

This yields

$$j^0 = \rho = \bar{\psi}\gamma^0\psi = \psi^+\psi \tag{9.8}$$

for the charge density and

$$\mathbf{j} = \bar{\psi}\boldsymbol{\gamma}\psi = \psi^+\boldsymbol{\alpha}\psi \tag{9.9}$$

for the current density. Continuity of current,

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \tag{9.10}$$

implies that the total field charge $Q = \int \rho d^3x$ is conserved and the integral (9.6) vanishes.

Thus, from the general expression for the field energy, (9.5), we obtain

$$P^0 = E = i\int d^3x\bar{\psi}\gamma^0\frac{\partial\psi}{\partial t} = i\int d^3x\psi^+\frac{\partial\psi}{\partial t}, \tag{9.11}$$

while the momentum is

$$\mathbf{P} = -i\int d^3x\bar{\psi}\gamma^0\nabla\psi = -i\int d^3x\psi^+\nabla\psi. \tag{9.12}$$

Note that the energy of the classical Dirac field is indefinite, that is, does not have a definite sign. At the same time, the field charge has a definite sign, it is nonnegative:

$$Q = \int \psi^+\psi\, d^3x = \int d^3x\,(\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4). \tag{9.13}$$

One of the main requirements imposed on a physically meaningful theory is that the energy of a free field be positive definite. For a Dirac field this requirement is satisfied only if the field undergoes quantization (see below).

Let us find the spin tensor for the field,

$$S^\nu_{\alpha\beta} = -\frac{\partial\mathscr{L}}{\partial\psi_{,\,\nu}}\mathrm{X}_{\alpha\beta}\psi - \bar{\psi}\bar{\mathrm{X}}_{\alpha\beta}\frac{\partial\mathscr{L}}{\partial\bar{\psi}_{,\,\nu}}, \tag{9.14}$$

where $\mathrm{X}_{\alpha\beta}$ and $\bar{\mathrm{X}}_{\alpha\beta}$ are the generators of the Lorentz group for ψ and $\bar{\psi}$, respectively:

$$\mathrm{X}_{\alpha\beta} = -\frac{i}{2}\sigma_{\alpha\beta}, \quad \bar{\mathrm{X}}_{\alpha\beta} = -\mathrm{X}_{\alpha\beta} = \frac{i}{2}\sigma_{\alpha\beta}. \tag{9.15}$$

The last formula follows from the definition of $\overline{X}_{\alpha\beta}$,

$$\overline{X}_{\alpha\beta} = \frac{i}{2}\gamma^0\sigma^+_{\alpha\beta}\gamma^0,$$

and the following property of the $\sigma_{\alpha\beta}$ matrix:

$$\overline{\sigma}_{\alpha\beta} = \gamma^0\sigma^+_{\alpha\beta}\gamma^0 = \sigma_{\alpha\beta}.$$

Substituting (9.15) into (9.14), we get

$$S^\nu_{\alpha\beta} = -\frac{1}{4}\overline{\psi}(\gamma^\nu\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\gamma^\nu)\psi. \tag{9.16}$$

Let us now take the integral of the zeroth component of the tensor (9.16),

$$S_{\alpha\beta} = \int d^3x S^0_{\beta\alpha}, \tag{9.17}$$

and find the total vector of the field spin S with components $S_1 = S_{23}$, $S_2 = S_{31}$, and $S_3 = S_{12}$ (an axial vector). Recalling that

$$\sigma_{ij} = \varepsilon_{ijk}\Sigma_k,$$

we arrive at the following explicit expression for the total spin of a Dirac field:

$$\mathbf{S} = \frac{1}{2}\int \psi^+\Sigma\psi\, d^3x, \tag{9.18}$$

where Σ is the Dirac matrix defined in (8.52). For a free Dirac field the spin by itself is not conserved. The integral of motion in this case is the total angular momentum

$$\mathbf{J} = \mathbf{S} + \mathbf{L}, \tag{9.19}$$

where **L** is the vector of the orbital angular momentum of the field, defined in (4.34) and (4.35).

9.2 The Massive Vector Field

The equation for the field functions u^μ that realize the vector representation of the Lorentz group can be written by analogy with the scalar case as

$$(\Box + m^2)u^\mu = 0 \quad (\mu = 0, 1, 2, 3). \tag{9.20}$$

It is clear that if $u^\mu(x)$ is transformed like a 4-vector, then Eq. (9.20) is invariant. But the equation implies that all four components of u^μ are independent of each other. What we need is the vector representation of the Lorentz group with three independent components of u corresponding to the vector representation of the group of three-dimensional rotations, that is, with the spin of the field equal

to unity. Note that in the case of a Dirac field we found that only two of the bispinor components out of four are independent at a fixed sign of the frequency. This corresponds to the spinor representation of the rotation group, that is, to a field with a spin equal to 1/2. Indeed, in the reference frame in which the Dirac particle is at rest, the spinor wave function has only two nonzero components, which correspond to the $SU(2)$ representation of the rotation group. In the same fashion, for a vector field only three components out of the four field components are independent. In the reference frame in which the vector particle is at rest only three components of the wave function are nonzero. In an arbitrary reference frame, due to Lorentz transformations, there appears a fourth component, which depends on the other three.

An additional condition that lowers the number of independent field components can be selected in the following form:

$$\frac{\partial}{\partial x^{\mu}} u^{\mu}(x) = 0. \tag{9.21}$$

This Lorentz-invariant equation is known as the Lorentz condition. In the case of a complex vector field, Eqs. (9.20) and (9.21) are augmented with the equations for the complex conjugate functions,

$$(\Box + m^2)u^{*\mu} = 0, \tag{9.22}$$

$$\partial_{\mu} u^{*\mu} = 0. \tag{9.23}$$

The additional conditions (9.21) and (9.23) can be obtained directly from the field equations if the Lagrangian density is chosen in the following form:

$$\mathscr{L} = -\frac{1}{2} f^{*\mu\nu} f_{\mu\nu} + m^2 u^{*\mu} u_{\mu}, \tag{9.24}$$

where the antisymmetric tensor $f_{\mu\nu}$ is

$$f_{\mu\nu} = \frac{\partial u_{\nu}}{\partial x^{\mu}} - \frac{\partial u_{\mu}}{\partial x^{\nu}}, \quad f_{\mu\nu} = -f_{\nu\mu}. \tag{9.25}$$

Combining the field equations

$$\frac{\partial \mathscr{L}}{\partial u^{\mu}} - \frac{\partial}{\partial x^{\nu}} \frac{\partial \mathscr{L}}{\partial u^{\mu}_{,\nu}} = 0$$

with the relationships

$$\frac{\partial \mathscr{L}}{\partial u_{\mu,\nu}} = f^{*\mu\nu}, \quad \frac{\partial \mathscr{L}}{\partial u^{*}_{\mu,\nu}} = f^{\mu\nu}, \tag{9.26}$$

which follow from (9.24), we obtain

$$m^2 u^\mu - \frac{\partial f^{\mu\nu}}{\partial x^\nu} = 0,$$
$$m^2 u^{*\mu} - \frac{\partial f^{*\mu\nu}}{\partial x^\nu} = 0. \tag{9.27}$$

These equations differ in appearance from Eqs. (9.20) and (9.22). However, if we take the 4-divergence of the left-hand sides of (9.27), we find

$$\frac{\partial u^\mu}{\partial x^\mu} = 0, \quad \frac{\partial u^{*\mu}}{\partial x^\mu} = 0, \tag{9.28}$$

that is, the Lorentz conditions as a corollary of field equations (9.27). Under these conditions, Eqs. (9.27) again take the form (9.20), (9.22).

Let us find the canonical energy-momentum tensor

$$T^\nu_\alpha = \frac{\partial \mathscr{L}}{\partial u_{\mu,\nu}} u_{\mu,\alpha} + \frac{\partial \mathscr{L}}{\partial u^*_{\mu,\nu}} u^*_{\mu,\alpha} - \mathscr{L}\delta^\nu_\alpha$$
$$= f^{*\mu\nu} u_{\mu,\alpha} + f^{\mu\nu} u^*_{\mu,\alpha} - \mathscr{L}\delta^\nu_\alpha. \tag{9.29}$$

It is not symmetric because the spin of the field is not zero. Let us now show that the energy of a vector field is positive definite. To this end we go over to canonical variables in the expression for the energy density

$$T^0_0 = f^{*\mu 0} u_{\mu,0} + f^{\mu 0} u^*_{\mu,0} - \mathscr{L}. \tag{9.30}$$

Note that

$$f^{*k0} = \frac{\partial \mathscr{L}}{\partial u_{k,0}} = \pi^k, \quad f^{k0} = \pi^{*k}, \quad f^{00} = f^{*00} = \pi^0 = \pi^{*0} = 0. \tag{9.31}$$

Moreover,

$$u_{k,0} = u_{0,k} - \pi^*_k,$$
$$u^0 = \frac{1}{m^2}\frac{\partial f^{0\nu}}{\partial x^\nu} = -\frac{1}{m^2}\frac{\partial \pi^{*k}}{\partial x^k}, \tag{9.32}$$
$$f^{*\mu\nu} f_{\mu\nu} = 2\pi^{*k}\pi_k + f^{*mk} f_{mk}.$$

Excluding, via (9.31) and (9.32), the time dependence of the derivatives from the expression (9.30) for T^0_0, we get

$$T^0_0 = -\pi^{*k}\pi_k + \frac{1}{2} f^{*mk} f_{mk} + \frac{1}{m^2}\frac{\partial \pi^{*k}}{\partial x^k}\frac{\partial \pi^m}{\partial x^m}$$
$$+ m^2 u^{*k} u^k + \frac{\partial}{\partial x^k}(\pi^k u_0 + \pi^{*k} u^*_0) \quad (k = 1, 2, 3). \tag{9.33}$$

Ignoring the last term with the divergence as contributing nothing to the total energy of the field and going over to three-dimensional notation, since the temporal components have been excluded, we arrive at

$$T_0^0 = \mathscr{H} = \boldsymbol{\pi}^* \cdot \boldsymbol{\pi} + \operatorname{curl} \mathbf{u}^* \cdot \operatorname{curl} \mathbf{u} + \frac{1}{m^2} \operatorname{div} \boldsymbol{\pi}^* \cdot \operatorname{div} \boldsymbol{\pi} + m^2 \mathbf{u}^* \cdot \mathbf{u}. \quad (9.34)$$

Note the following special features of this expression for T_0^0. Firstly, (9.34) depends only on the canonical variables $\mathbf{u}$, $\mathbf{u}^*$, $\boldsymbol{\pi}$, $\boldsymbol{\pi}^*$ and their spatial derivatives. Thus, (9.34) defines the Hamiltonian density of the vector field. Next, in the vector case the energy density $\mathscr{H}$ is nonnegative, in contrast to the Dirac case. Finally, (9.24) is meaningful only under the assumption that the mass of the field, m, is nonzero. In the limit of a massless field, $m \to 0$, the expression (9.34) is divergent, and this means that the case of a massless field requires special investigation (see below). A complex vector field possesses a charge. The 4-vector current density is

$$j^\mu = -i \left(u^\sigma \frac{\partial \mathscr{L}}{\partial u^\sigma_{,\mu}} - u^{*\sigma} \frac{\partial \mathscr{L}}{\partial u^{*\sigma}_{,\mu}} \right) = -i (u_\sigma f^{*\sigma\mu} - u^*_\sigma f^{\sigma\mu}). \quad (9.35)$$

From this it follows that the field charge is

$$Q = \int j^0 d^3x = -i \int d^3x \, (u_k \pi^k - u^*_k \pi^{*k}). \quad (9.36)$$

The total charge of a vector field has no definite sign, just as in the case of a scalar field, and a meaningful interpretation of this quantity may be achieved only through the second quantization technique.

The spin of a vector field is nonzero:

$$S^\nu_{\alpha\beta} = -u_\sigma (X_{\alpha\beta})^\sigma_\rho \frac{\partial \mathscr{L}}{\partial u_{\rho,\nu}} + \text{c.c.} = -(X_{\alpha\beta})^\sigma_\rho (u_\sigma f^{*\rho\nu} + u^*_\sigma f^{\rho\nu}). \quad (9.37)$$

Using the expression for the generator of the Lorentz group,

$$(X_{\alpha\beta})^\sigma_\rho = \delta^\sigma_\alpha g_{\beta\rho} - \delta^\sigma_\beta g_{\alpha\rho},$$

we find that

$$S^\nu_{\alpha\beta} = -u_\alpha f^{*\nu}_\beta + u_\beta f^{*\nu}_\alpha + \text{c.c.} \quad (9.38)$$

The total vector of the spin of the field is related to the total tensor of the spin

$$S_{\alpha\beta} = \int S^0_{\beta\alpha} d^3x = \int d^3x \, (u_\alpha \pi_\beta - u_\beta \pi_\alpha + \text{c.c.}) \quad (9.39)$$

thus:

$$S_i = \frac{1}{2} e_{ikj} S_{kj} \quad (i = 1, 2, 3). \quad (9.40)$$

This yields the following formula in the three-dimensional notation:

$$\mathbf{S} = \int d^3x\,(\mathbf{u} \times \boldsymbol{\pi} + \mathbf{u}^* \times \boldsymbol{\pi}^*). \tag{9.41}$$

10. The Electromagnetic Field

The fields we considered up till now, the scalar field, the Dirac field, and the massive vector field, were introduced to a certain extent in a formal manner, as objects of the appropriate representations of the Lorentz group. In this section we consider a special case of a vector field, the massless vector field. The corresponding real physical field is the well-known electromagnetic field. We will show that this emerges in a natural way from the requirement that the theory of charged fields be local gauge invariant.

10.1 Local Gauge Transformations

In Section 6 we became acquainted with gauge transformations of the form

$$u^A(x) \to u'^A(x) = e^{-i\alpha} u^A(x), \quad u^{*A}(x) \to u^{*\prime A}(x) = e^{i\alpha} u^{*A}(x), \tag{10.1}$$

where the real parameter α is assumed to be independent of coordinates x. Such a transformation is called global, since it is carried out in exactly the same manner at all points in space-time.

Let us now assume that the gauge transformation (10.1) is carried out differently at different points x, that is, parameter α is a function of the coordinates of the points in space-time:

$$\alpha = \alpha\,(x). \tag{10.2}$$

Such a transformation is called a local gauge transformation, or a gauge transformation of the second kind.

Next we will require that physical processes be invariant under local gauge transformations, that is, that the Lagrangian density be actually an invariant in the Lagrange theory. We take a free Dirac field specified by the Lagrangian density

$$\mathcal{L}_0 = \frac{i}{2}(\bar{\psi}\gamma\,\partial\psi - \partial\bar{\psi}\gamma\psi) - m\bar{\psi}\psi. \tag{10.3}$$

Suppose that

$$\psi\,(x) \to e^{-i\alpha(x)}\psi\,(x) = \psi'\,(x), \quad \bar{\psi}\,(x) \to e^{i\alpha(x)}\bar{\psi}\,(x) = \bar{\psi}'\,(x), \tag{10.4}$$

where α is a scalar function of $x = (t, \mathbf{x})$. Then (10.3) becomes

$$\begin{aligned}\mathcal{L}'_0(\psi', \bar{\psi}') &= \frac{i}{2}\{\bar{\psi}\gamma^\mu(\partial_\mu - i\alpha_{,\mu})\cdot\psi - (\partial_\mu\bar{\psi} + i\alpha_{,\mu}\bar{\psi})\,\gamma^\mu\psi\} - m\bar{\psi}\psi \\ &= \mathcal{L}_0(\psi, \bar{\psi}) + \alpha_{,\mu}\bar{\psi}\gamma^\mu\psi.\end{aligned} \tag{10.5}$$

As the reader can see, the free Lagrangian density is not an invariant in this case. To compensate for the noninvariant term $\alpha_{,\mu}\bar{\psi}\gamma^{\mu}\psi$, we introduce an auxiliary vector field $A_{\mu}(x)$ (the gauge, or compensating, field) and instead of $\mathcal{L}_0$ write the following Dirac Lagrangian density:

$$\mathcal{L}_{\mathrm{D}}=\frac{i}{2}(\bar{\psi}\gamma_{\mu}D^{\mu}\psi-(D_{\mu}^{*}\bar{\psi})\gamma^{\mu}\psi)-m\bar{\psi}\psi, \tag{10.6}$$

where instead of the derivative $\partial_{\mu}=\partial/\partial x^{\mu}$ we have introduced the "extended" derivative

$$D^{\mu}=\partial^{\mu}+ieA^{\mu}, \tag{10.7}$$

in which A^{μ} is the vector field, and e is the coupling constant that links the complex field ψ with the real field A^{μ} (the charge of the Dirac field).

The Lagrangian density (10.6) will be local gauge invariant if, together with the Dirac field (10.4), the gauge field A^{μ} is transformed in such a manner that under the transformations of ψ and $\bar{\psi}$ the extended derivatives $D\psi$ and $D^{*}\bar{\psi}$ are transformed according to the same law:

$$D_{\mu}\psi\to D'_{\mu}\psi'=e^{-i\alpha}D_{\mu}\psi,\quad D_{\mu}^{*}\bar{\psi}\to D_{\mu}'^{*}\bar{\psi}'=e^{i\alpha}D_{\mu}^{*}\bar{\psi}. \tag{10.8}$$

For this to be true, it is sufficient to require that the gauge transformation of the field $A_{\mu}(x)$, carried out simultaneously with the transformation of ψ and $\bar{\psi}$, have the form of a gradient transformation

$$A_{\mu}(x)\to A'_{\mu}(x)=A_{\mu}(x)+\frac{1}{e}\partial_{\mu}\alpha. \tag{10.9}$$

Then

$$\begin{aligned}(D_{\mu}\psi)'&=(\partial_{\mu}+ieA'_{\mu})\psi'=(\partial_{\mu}+ieA_{\mu}+i\partial_{\mu}\alpha)e^{-i\alpha}\psi\\&=e^{-i\alpha}(\partial_{\mu}+ieA_{\mu}-i\partial_{\mu}\alpha+i\partial_{\mu}\alpha)\psi=e^{-i\alpha}D_{\mu}\psi\end{aligned}$$

and similarly for $D_{\mu}^{*}\bar{\psi}$, which ensures the invariance of the Lagrangian density (10.6):

$$\mathcal{L}_{\mathrm{D}}(\psi,\ \bar{\psi},\quad A)=\mathcal{L}_{\mathrm{D}}(\psi',\ \bar{\psi}',\ A').$$

The new vector field $A_{\mu}(x)$, which was introduced for the sake of local gauge invariance, is in itself a dynamical system. This requires writing out the Lagrangian density and the dynamical equations, the field equations that this field must obey, in addition to the Lagrangian density and the field equations for the Dirac field.

We write the Lagrangian for A_μ in the same way as we did in Section 9 for the massive vector field:

$$\mathcal{L}_f = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}, \tag{10.10}$$

where the field-strength tensor

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \tag{10.11}$$

is obviously invariant under transformation (10.9):

$$F'_{\mu\nu} = \partial_\mu \left(A_\nu + \frac{1}{e}\partial_\nu \alpha\right) - \partial_\nu \left(A_\mu + \frac{1}{e}\partial_\mu \alpha\right) = F_{\mu\nu}.$$

The mass term, proportional to $A_\mu A^\mu$, is not present in (10.10), since its presence would break the gauge invariance of $\mathcal{L}_f$.

The Lagrangian density given by (10.10) describes a massless vector gauge field, which can be identified with the well-known electromagnetic, or Maxwellian, field.

As usual, the components of the electric field **E** and the magnetic field **H** will be related to the components of the antisymmetric field-strength tensor in the following manner:

$$E_k = F_{0k}, \quad H_k = -\frac{1}{2}\varepsilon_{kln}F^{ln}, \tag{10.11a}$$

or, explicitly,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}. \tag{10.11b}$$

To describe a system consisting of a charged Dirac field and an electromagnetic field, we must write the total Lagrangian as the sum of the Dirac Lagrangian (10.6) and the electromagnetic-field Lagrangian (10.10):

$$\mathcal{L} = \mathcal{L}_D + \mathcal{L}_f = \frac{i}{2}(\bar{\psi}\gamma D\psi - D^*\bar{\psi}\gamma\psi) - m\bar{\psi}\psi - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}. \tag{10.12}$$

If in this expression we isolate the Lagrangian $\mathcal{L}_0$ of the free Dirac field, (10.3), then $\mathcal{L}$ can be represented by a sum of three terms:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_f + \mathcal{L}_{\text{int}}, \tag{10.13}$$

where

$$\mathcal{L}_{\text{int}} = -eA_\mu\bar{\psi}\gamma^\mu\psi = -eA_\mu j^\mu. \tag{10.14}$$

In the total Lagrangian this term describes the interaction of the charged field ψ and the electromagnetic field A_μ and is called the interaction Lagrangian. The other two terms in $\mathscr{L}$ describe the free Dirac field, $\mathscr{L}_0$, and the free electromagnetic field, $\mathscr{L}_f$.

Note that by replacing ∂_μ with D_μ we can build, in a quite similar manner, a gauge invariant Lagrangian of a charged scalar field.

The gauge invariant Lagrangian (10.12), constructed by replacing ∂_μ with $\partial_\mu + ieA_\mu$ in $\mathscr{L}_0$, takes into account the interaction with the electromagnetic field in the so-called minimal way. Generally speaking, a gauge invariant term of the form

$$\Delta\mu\bar{\psi}\sigma_{\alpha\beta}\psi F^{\alpha\beta} \tag{10.15}$$

is also possible; it describes the interaction of the so-called anomalous magnetic moment $\Delta\mu$ directly with the field-strength tensor $F^{\alpha\beta}$. This type of interaction, introduced by W. Pauli, requires another constant $\Delta\mu$ in addition to charge e. However, for electrons we can restrict our discussion to one constant, e, in full agreement with the experimental data.* Terms of the (10.15) type appear only in the effective Lagrangians of the nucleons, n and p, due to the common electromagnetic interaction with the cloud of virtual pions surrounding a nucleon.

10.2 Electromagnetic-Field Equations

We can now use Lagrangian (10.12) to write the equations for the fields ψ and A_μ. For the Dirac field, from the general Lagrange-Euler equation

$$\frac{\partial\mathscr{L}}{\partial\bar{\psi}} - \frac{\partial}{\partial x^\mu}\frac{\partial\mathscr{L}}{\partial\bar{\psi}_{,\mu}} = 0$$

we get the Dirac equation incorporating the electromagnetic field,

$$[(i\partial_\mu - eA_\mu)\gamma^\mu - m]\psi = 0, \tag{10.16}$$

which differs from the equation for a free Dirac field by the substitution of D_μ for ∂_μ.

The equations for the vector field A_μ, the 4-potential of the electromagnetic field,

$$\frac{\partial\mathscr{L}}{\partial A_\mu} - \frac{\partial}{\partial x^\nu}\frac{\partial\mathscr{L}}{\partial A_{\mu,\nu}} = 0$$

* Note that the known anomalous magnetic moment of the electron is purely a vacuum effect and emerges as a result of the nonlinear effective interaction with the vacuum at minimal initial interaction (see Section 35).

combined with

$$\frac{\partial \mathscr{L}}{\partial A_\mu} = -ej^\mu, \quad j^\mu = \bar{\psi}\gamma^\mu\psi, \quad \frac{\partial \mathscr{L}}{\partial A_{\mu,\nu}} = \frac{1}{4\pi} F^{\mu\nu} \tag{10.17}$$

yield

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = -4\pi e j^\mu. \tag{10.18}$$

These are the well-known Maxwell equations for the electromagnetic field. Note that the electromagnetic current in the term on the right-hand side of (10.18) satisfies, in view of the fact that $F^{\mu\nu}$ is an antisymmetric tensor, the continuity equation

$$\frac{\partial j^\mu}{\partial x^\mu} = 0, \tag{10.18a}$$

which expresses the (global) gauge invariance of the theory. The current is a quadratic combination of Dirac fields, $j^\mu = \bar{\psi}\gamma^\mu\psi$, which obey Eq. (10.16), and this equation also contains the 4-potential A_μ. Equations (10.16) and (10.18) comprise a system of nonlinear equations for interacting Dirac and Maxwellian fields. The system may be solved only if we introduce assumptions that simplify it considerably. In particular, it is customary to fix the electromagnetic current as a function of coordinates and time, and then we arrive either at problems solved by methods of classical electrodynamics (see Chapter II) or at the problem of radiation by a classical current in quantum electrodynamics (see Chapter III), or to fix the electromagnetic field as a function of coordinates and time, and then we arrive at the problem of describing the Dirac field placed in an external electromagnetic field (see Chapter IV). The problem of a Dirac field and an electromagnetic field moving like a coupled dynamical system can be solved in general form only approximately by employing the quantum perturbation theory (see Chapter III).

We write the equation for potential A_μ

$$\Box A_\mu - \frac{\partial^2 A_\nu}{\partial x^\mu \partial x_\nu} = 4\pi e j_\mu. \tag{10.19}$$

To simplify this equation, we use the fact that the electromagnetic field is gradient invariant (note that the terms gradient invariance and local gauge invariance mean the same thing). We begin with a potential A'_μ for which

$$\frac{\partial A'_\mu}{\partial x_\mu} \neq 0.$$

Let us now select the gauge function α in such a manner that the potential $A_\mu = A'_\mu + 1/e\partial_\mu\alpha$ satisfies the Lorentz condition

$$\frac{\partial A_\mu}{\partial x_\mu} = \frac{\partial}{\partial x_\mu}\left(A'_\mu + \frac{1}{e}\frac{\partial\alpha}{\partial x^\mu}\right) = 0. \tag{10.20}$$

The sufficient condition for this is

$$\Box\alpha = -e\frac{\partial A'^\mu}{\partial x^\mu}. \tag{10.21}$$

Under condition (10.20), Eq. (10.19) becomes an inhomogeneous d'Alembert equation

$$\Box A_\mu = 4\pi e j_\mu, \tag{10.22}$$

on whose right-hand side we have the electromagnetic current. The function α is defined by Eq. (10.21) in a nonunique fashion, precisely, to within a solution of the homogeneous d'Alembert equation:

$$\Box\alpha_0 = 0. \tag{10.23}$$

For this reason the potential A_μ, which obeys Eq. (10.22), allows for an additional gauge transformation with a gauge function α_0 satisfying Eq. (10.23).

10.3 Coulomb Gauge

Let us put $j^0 = 0$ and $\mathbf{j} = 0$ and consider the free electromagnetic field. Then we can select a gauge function α_0 such that in a specially selected reference frame the temporal component of the potential is absent, $A^0 = 0$, and instead of (10.20) we get the three-dimensional condition

$$\operatorname{div}\mathbf{A} = 0. \tag{10.24}$$

This combined with the fact that $A^0 = 0$ implies that from Eq. (10.19) we can go over to d'Alembert's equation for the vector potential $\mathbf{A}$:

$$\Box\mathbf{A} = 0. \tag{10.25}$$

Gauge (10.24) is known as Coulomb gauge. If we write the vector potential in the form of a Fourier integral,

$$\mathbf{A} = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{A}(\mathbf{k})\,d^3k, \tag{10.26}$$

then condition (10.24) leads to

$$\mathbf{k}\cdot\mathbf{A}(\mathbf{k}) = 0, \tag{10.27}$$

which simply means that the Fourier transform of the vector potential, $\mathbf{A}(\mathbf{k})$, is orthogonal to the wave vector $\mathbf{k}$, whereby Coulomb

gauge is sometimes called three-dimensional transverse gauge. Coulomb gauge is not relativistically invariant, that is, condition (10.24) is violated if we go over to another frame of reference. But it provides the easiest means for demonstrating the fact that the energy of the field is positive definite and for finding the Hamiltonian of the field.

Note that we cannot use the method of Section 9 for $m \neq 0$ and then proceed to the limit by sending m to zero (the reasons for this were discussed there, too). For this reason we will go over to canonical variables directly for the case of the massless electromagnetic field. In Coulomb gauge we have, actually, a three-dimensional problem, so that the Hamiltonian density depends on the three-component vectors π and $\mathbf{A}$ and their spatial derivatives:

$$\mathscr{H} = \mathscr{H}(\pi, \mathbf{A}, A_{k,t}). \tag{10.28}$$

To find the explicit form of $\mathscr{H}$ we write the Lagrangian density in terms of the electric and magnetic fields (10.11a):

$$\mathscr{L} = \frac{\mathbf{E}^2 - \mathbf{H}^2}{8\pi}, \tag{10.29}$$

where in the case at hand $\mathbf{E} = -\partial\mathbf{A}/\partial t$ and $\mathbf{H} = \operatorname{curl}\mathbf{A}$. Recalling the definition of the density of the canonical momentum π given in Section 1, we write

$$\pi = \frac{\partial\mathscr{L}}{\partial\mathbf{A}_{,t}} = -\frac{\mathbf{E}}{4\pi}, \tag{10.30}$$

and in view of the field equations (10.18) without currents, $\operatorname{div}\mathbf{E} = 0$ and, hence, $\operatorname{div}\pi = 0$.

Let us now calculate the Hamiltonian density $\mathscr{H}$, which is equal to the component T_0^0 of the energy-momentum tensor, that is, the energy density of the field:

$$\mathscr{H} = \frac{\partial\mathscr{L}}{\partial\mathbf{A}_{,t}}\mathbf{A}_{,t} - \mathscr{L} = \frac{\mathbf{E}^2}{4\pi} - \frac{\mathbf{E}^2 - \mathbf{H}^2}{8\pi} = \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi}. \tag{10.31}$$

We rewrite this expression by isolating the explicit functional dependence of $\mathscr{H}$ on the canonical variables:

$$\mathscr{H} = \frac{(4\pi\pi)^2 + (\operatorname{curl}\mathbf{A})^2}{8\pi}. \tag{10.32}$$

Thus, just like in the case of a massive field, we arrive at a positive definite expression for the energy ($\mathscr{H} \geqslant 0$). Note, however, that in the massive case the condition $\partial_\mu A^\mu = 0$, which ensured that $\mathscr{H} \geqslant 0$, followed automatically from the Lagrangian of the theory, while in the massless case we ensure that the field energy has a definite sign ($\mathscr{H} \geqslant 0$) thanks to gauge invariance and to an appropriate choice of gauge, a choice that eliminates the temporal component, $A^0 = 0$, and restricts $\mathbf{A}$ to the transversality condition $\operatorname{div}\mathbf{A} = 0$.

It can easily be verified that the Hamilton equations, which can be obtained through the function (10.32), lead to the common Maxwell equations in three-dimensional form for a free field ($\mathbf{j} = 0$, $\rho = 0$)

$$\operatorname{div} \mathbf{E} = 0, \quad \operatorname{curl} \mathbf{H} - \partial \mathbf{E}/\partial t = 0. \tag{10.33}$$

The same equations follow from Eq. (10.18). The other pair of the Maxwell equations,

$$\operatorname{div} \mathbf{H} = 0, \quad \operatorname{curl} \mathbf{E} + \partial \mathbf{H}/\partial t = 0, \tag{10.34}$$

in the given gauge are satisfied automatically, since

$$\operatorname{div} \mathbf{H} = \operatorname{div} \operatorname{curl} \mathbf{A} \equiv 0, \tag{10.35}$$

$$\operatorname{curl} \mathbf{E} + \partial \mathbf{H}/\partial t = \operatorname{curl} (-\partial \mathbf{A}/\partial t) + \partial (\operatorname{curl} \mathbf{A})/\partial t \equiv 0.$$

In the general case the second pair of the Maxwell equations, (10.34), can be obtained as a result of defining $F_{\mu\nu}$ in terms of the potentials. Indeed, in four-dimensional form, (10.34) is

$$\frac{\partial}{\partial x_\mu} \tilde{F}_{\mu\nu} = 0, \tag{10.36}$$

where

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\lambda\rho} F^{\lambda\rho} \tag{10.37}$$

is the dual field-strength tensor, with $\varepsilon_{\mu\nu\lambda\rho}$ the totally antisymmetric fourth-rank tensor normalized by the condition $\varepsilon_{0123} = -1$. The components of tensor $\tilde{F}_{\mu\nu}$ are related to the electric and magnetic field components thus:

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & H_x & H_y & H_z \\ -H_x & 0 & E_z & -E_y \\ -H_y & -E_z & 0 & E_x \\ -H_z & E_y & -E_x & 0 \end{pmatrix}. \tag{10.38}$$

Equation (10.36) will become an identity if we substitute the definition of the field-strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Indeed,

$$\frac{\partial}{\partial x^\mu} \varepsilon^{\mu\nu\lambda\rho} (\partial_\lambda A_\rho - \partial_\rho A_\lambda) = 2\varepsilon^{\mu\nu\lambda\rho} \partial_\mu \partial_\lambda A_\rho \equiv 0, \tag{10.39}$$

since contraction of a symmetric tensor $\partial_\mu \partial_\lambda A_\rho$ (in μ and λ) with the antisymmetric tensor $\varepsilon^{\mu\nu\lambda\rho}$ always yields zero.

Thus, the requirement of gauge invariance leads to the conservation of electromagnetic current and to the necessity of introducing an electromagnetic field that serves as the agent through which charged fields interact. The role of the electromagnetic field is twofold: it is the gauge field and the interaction agent between the

charges. Another result of gauge invariance is the masslessness, so to say, of the electromagnetic field, as a result of which we arrive at the transversality of the free electromagnetic field in the Coulomb gauge, in which only two out of the four components A^μ remain independent ($A^0 = 0$, div $\mathbf{A} = 0$).

10.4 Lorentz Gauge

The construction of a Hamiltonian theory and the fact that the energy of the electromagnetic field is positive definite are the necessary prerequisites for quantization of the field. But, as we know, Coulomb gauge is not relativistically invariant. For a covariant generalization of the theory it has proved inconvenient to use the Lagrangian (10.10), since the method of eliminating the temporal component A^0 and the time derivatives discussed in Section 9 in connection with a massive field cannot be applied to a massless field. For this reason we introduce the gauge noninvariant Lagrangian

$$\mathscr{L}_f = -\frac{1}{8\pi} A_{\mu,\,\nu} A^{\mu,\,\nu} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{8\pi} A_{\mu,\,\nu} A^{\nu,\,\mu}. \quad (10.40)$$

The field equations obtained via this Lagrangian have the form

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial A^\nu}{\partial x^\nu}\right) = 0. \quad (10.41)$$

For the potentials we obtain d'Alembert's equations

$$\Box A_\mu = 0 \ (\mu = 0,\ 1,\ 2,\ 3). \quad (10.42)$$

The divergence of potentials, $\partial A^\mu/\partial x^\mu$, generally has an arbitrary value (the Lagrangian and the field equations do not imply that the divergence must be zero, in contrast to the case of a massive field). For this reason, Eqs. (10.41) are not identical to the Maxwell equations. Suppose at the initial moment, $t = 0$, the divergence $A^\mu_{,\mu}$ satisfies the conditions

$$A^\mu_{,\mu}|_{t=0} = 0, \quad \frac{\partial}{\partial t} A^\mu_{,\mu}|_{t=0} = 0. \quad (10.43)$$

Since the divergence $A^\mu_{,\mu}$ satisfies d'Alembert's equation

$$\Box A^\mu_{,\mu} = 0, \quad (10.44)$$

in view of (10.43), at an arbitrary moment it will also be zero:

$$\frac{\partial A^\mu}{\partial x^\mu} = 0. \quad (10.45)$$

This proves that by properly selecting the initial conditions, (10.43), we can always be sure that the Lorentz condition (10.45) for the poten-

tials is satisfied. Equations (10.41) then simply transform into the Maxwell equations:

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = 0. \tag{10.46}$$

Thus, out of all of the more general fields that obey d'Alembert's equation (10.42) the Lorentz condition (10.45) selects only the Maxwellian fields.

Let us now require that the Lorentz condition (10.45) be invariant under gradient (or local gauge) transformations

$$A^{\mu} \to A'^{\mu} = A^{\mu} + \frac{1}{e}\frac{\partial \alpha}{\partial x_{\mu}}. \tag{10.47}$$

Then the condition

$$\frac{\partial A'^{\mu}}{\partial x^{\mu}} = \frac{\partial A^{\mu}}{\partial x^{\mu}} + \frac{1}{e}\frac{\partial^2 \alpha}{\partial x_{\mu}\,\partial x^{\mu}} = 0$$

yields the following equation:

$$\Box\alpha = 0. \tag{10.48}$$

This condition, which restricts the choice of gauge function α, also restricts the possible gauges of the potential. It has been found that under gauge transformations for which $\Box\alpha = 0$ the Lagrangian (10.48) is gauge invariant to within a 4-divergence. Indeed, in the transition $A_{\mu} \to A'_{\mu}$ given by (10.47), the Lagrangian (10.40) becomes

$$\mathscr{L}(A') = -\frac{1}{8\pi} A_{\mu,\,\nu} A^{\mu,\,\nu} - \frac{1}{8\pi e^2}(\alpha_{,\,\mu\nu} - 2eA_{\mu,\,\nu})\,\alpha^{,\,\mu\nu}. \tag{10.49}$$

Transforming the last term on the right-hand side with due regard for (10.48), we find that

$$(\alpha_{,\,\mu\nu} - 2eA_{\mu,\,\nu})\,\alpha^{,\,\mu\nu} = -\frac{\partial}{\partial x^{\nu}}[(2eA_{\mu} - \alpha_{,\,\mu})\,\alpha^{,\,\mu\nu}],$$

which simply means that this term can be dropped without any effect on the field equations.

Let us now find the energy density of the electromagnetic field

$$\begin{aligned} T_0^0 &= A_{\mu,\,0}\frac{\partial \mathscr{L}}{\partial A_{\mu,\,0}} - \mathscr{L} = -\frac{1}{8\pi}(A_{\mu,\,0}A^{\mu,\,0} - A_{\mu,\,k}A^{\mu,\,k}) \\ &= -\frac{1}{8\pi}\left(\frac{\partial A^0}{\partial t}\right)^2 + \frac{1}{8\pi}\left(\frac{\partial \mathbf{A}}{\partial t}\right)^2 - \frac{1}{8\pi}\left(\frac{\partial A^0}{\partial \mathbf{x}}\right)^2 + \frac{1}{8\pi}A_{k,\,l}A^{k,\,l}. \end{aligned} \tag{10.50}$$

This expression has no definite sign, but it can be transformed in such a manner that to within a three-dimensional divergence the result is positive definite. Let us add the three-dimensional divergence

$$\frac{1}{4\pi}\frac{\partial}{\partial x^k}\left(A^0\frac{\partial A^k}{\partial x_0}\right)=\frac{1}{4\pi}\left(\frac{\partial A_0}{\partial x^k}\frac{\partial A^k}{\partial x_0}+A_0\frac{\partial}{\partial x^0}\frac{\partial A^k}{\partial x^k}\right)$$

to (10.50). The second term on the right-hand side of this expression can be written, to within a divergence, in the form

$$\frac{1}{4\pi}\left(\frac{\partial A^0}{\partial x^k}\frac{\partial A^k}{\partial t}+\left(\frac{\partial A^0}{\partial x^k}\right)^2\right),$$

where we have allowed for the Lorentz condition and the equation $\Box A^0 = 0$. Then (10.50) takes the form

$$T_0^0=-\frac{1}{8\pi}\left(\frac{\partial A^0}{\partial t}\right)^2+\frac{1}{8\pi}\left(\frac{\partial A^0}{\partial x^k}+\frac{\partial A^k}{\partial t}\right)^2=\frac{1}{8\pi}A_{k,\,l}A^{k,\,l}.$$

We use the identities

$$A_{k,\,l}A^{k,\,l}-A_{k,\,l}A^{l,\,k}=\frac{1}{2}F_{kl}F^{kl}=(\operatorname{curl}\mathbf{A})^2,$$

$$A_{k,\,l}A^{l,\,k}=\frac{\partial}{\partial x^l}\left(A_k\frac{\partial A^l}{\partial x_k}-A^l\frac{\partial A^k}{\partial x^k}\right)+\frac{\partial A^k}{\partial x^k}\frac{\partial A^l}{\partial x^l}.$$

Then discarding the divergence, we get

$$T_0^0=-\frac{1}{8\pi}\left(\frac{\partial A^0}{\partial t}\right)^2+\frac{1}{8\pi}\left(\frac{\partial A^0}{\partial \mathbf{x}}+\frac{\partial \mathbf{A}}{\partial t}\right)^2$$
$$+\frac{1}{8\pi}(\operatorname{curl}\mathbf{A})^2+\frac{1}{8\pi}(\operatorname{div}\mathbf{A})^2. \qquad (10.51)$$

The first term in this formula yields a negative contribution to the energy, but thanks to the Lorentz condition it is compensated by the last term. The first term is temporal and was absent in the Coulomb gauge, $A_0 = 0$. The last term is longitudinal and in the Coulomb gauge was absent, too, $\operatorname{div}\mathbf{A} = 0$. As we see, in the Lorentz gauge the sum of these terms is zero. The result is a positive definite expression:

$$T_0^0=\frac{1}{8\pi}\left[(\operatorname{curl}\mathbf{A})^2+\left(\frac{\partial \mathbf{A}}{\partial t}+\nabla A^0\right)^2\right]=\frac{1}{8\pi}(\mathbf{H}^2+\mathbf{E}^2), \qquad (10.52)$$

where we have used the definitions for the magnetic field $\mathbf{H} = \operatorname{curl}\mathbf{A}$ and the electric field $\mathbf{E} = -d\mathbf{A}/\partial t - \nabla A^0$. Thus, the final expressions for T_0^0 in the Coulomb and Lorentz gauges coincide, since the values of $\mathbf{E}$ and $\mathbf{H}$ do not depend on the choice of gauge. Divergence contributes nothing to the total integral over volume and does not change the total energy of the field.

We introduce the density of the canonical momentum thus:

$$\pi_\mu = \frac{\partial \mathscr{L}}{\partial A^{\mu,0}} = -\frac{1}{4\pi} A_{\mu,0}, \quad \pi = \frac{1}{4\pi}\frac{\partial \mathbf{A}}{\partial t}. \tag{10.53}$$

Then we can write $T^0_0 = \mathscr{H}$ as a function of the canonical variables:

$$T^0_0 = \mathscr{H}(A^\mu_{,k}, \pi^k) = \frac{1}{8\pi}[(\text{curl } \mathbf{A})^2 + (4\pi\pi)^2 + 8\pi(\pi \cdot \nabla A^0) + (\nabla A^0)^2]. \tag{10.54}$$

We have thus established that Lorentz gauge, which is relativistically invariant, ensures that the energy of the field is positive definite and that the Hamiltonian method can be employed. The expressions for the energy density in Coulomb gauge and Lorentz gauge coincide, to within a divergence.

11. Gauge Fields

As shown in Section 10, the electromagnetic field is a gauge field. It ensures that charged fields are local gauge invariant, since it obeys a definite rule of gauge transformations, compensating for the noninvariant terms that appear under transformations of a free charged field. What we considered was actually a local group of gauges specified by a single gauge function, a one-parameter group. C.N. Yang and R.L. Mills carried out a local generalization of the three-parameter rotation group. In their studies they arrived at the necessity of introducing three-component gauge fields, now known as Yang-Mills fields. Lately the concept of Yang-Mills fields and its generalization have occupied an important place in theoretical studies of elementary particle interactions.

11.1 An Arbitrary Gauge Group

Let us consider a gauge group

$$\psi(x) \to \psi'(x) = e^{-i\theta^a L_a}\psi(x) = \omega\psi(x), \tag{11.1}$$

where $a = 1, 2, \ldots, n$, with n the number of the parameters in the group. Suppose that the parameters $\theta^a = \theta^a(x)$ are functions of coordinates. The group becomes local. The hermitian generators L_a of the group obey the commutation relations

$$[L_a, L_b] = ic_{abc}L_c, \tag{11.2}$$

with c_{abc} the structure constants of the group. Obviously, the transformation $\omega(x) = \exp(-i\theta^a L_a)$ is unitary:

$$\omega^+ = \omega^{-1} = \exp(i\theta^a L_a).$$

The derivative $\partial_\mu \psi(x)$ does not obey the transformation rule (11.1) for the functions $\psi(x)$, since the transformation is local, $\omega = \omega(x)$ and $\partial_\mu \omega \neq 0$. Therefore, in a manner similar to that employed in the case of a one-parameter group in Section 10, instead of ∂_μ we introduce the generalized derivative D_μ so that $D_\mu \psi$ obeys the transformation law (11.1):

$$D_\mu \psi(x) \to D'_\mu \psi'(x) = \omega(x) D_\mu \psi(x). \tag{11.3}$$

Then

$$(D_\mu \psi)' = \omega D_\mu \omega^{-1} \omega \psi = \omega D_\mu \omega^{-1} \psi' = D'_\mu \psi',$$

whence

$$D'_\mu = \omega D_\mu \omega^{-1}. \tag{11.4}$$

This condition can be met if we select D_μ in the form

$$D_\mu = I\partial_\mu - ieW_\mu, \tag{11.5}$$

where I is the unit operator, W_μ is an operator function, called the potential of the gauge field,

$$W_\mu = W^a_\mu L_a, \tag{11.6}$$

and e is the coupling constant linking the fields ψ and W^a_μ. Then*

$$D'_\mu = \omega D_\mu \omega^{-1} = \omega(\partial_\mu - ieW_\mu)\omega^{-1} = \partial_\mu - ieW'_\mu,$$

from which we obtain the transformation law for the potential,

$$W'_\mu = \omega W_\mu \omega^{-1} - \frac{1}{e}\partial_\mu \omega \omega^{-1}. \tag{11.7}$$

Here we have employed the identity

$$\omega \partial_\mu \omega^{-1} + \partial_\mu \omega \omega^{-1} = 0, \tag{11.7a}$$

which follows from the condition $\omega\omega^{-1} = 1$.

The gauge field introduced in this manner makes it possible to satisfy condition (11.3) and ensure the invariance of the theory under transformations belonging to the gauge group.

Let us consider the infinitesimal transformation

$$\omega = I - i\theta^a L_a. \tag{11.8}$$

Then

$$W'_\mu = W_\mu - i\theta^a W^b_\mu [L_a, L_b] - \frac{1}{e}\partial_\mu \theta^a L_a.$$

* We will no longer write the unit operator I, assuming its presence where it is necessary.

Using the commutation relations (11.2) and expansion (11.6), we arrive at the following formula for the components of the potential:

$$W'^{c}_{\mu} = W^{c}_{\mu} + c_{abc}\theta^{a}W^{b}_{\mu} - \frac{1}{e}\partial_{\mu}\theta^{c}. \tag{11.9}$$

We see that this transformation contains, aside from the inhomogeneous gradient term $-(1/e)\,\partial_{\mu}\theta^{c}$ that exists in the gauge transformation in electrodynamics, a homogeneous term $c_{abc}\theta^{a}W^{b}_{\mu}$. This term is characteristic of a non-Abelian theory with noncommutative potentials, $[W_{\mu}, W_{\nu}] \neq 0$. In a one-parameter (Abelian) theory such as electrodynamics this term simply does not exist.

We will now define the field-strength tensor in the following manner:

$$G_{\mu\nu} = D_{\mu}W_{\mu} - D_{\nu}W_{\mu} = \partial_{\mu}W_{\nu} - \partial_{\nu}W_{\mu} + \frac{e}{i}[W_{\mu}, W_{\nu}]. \tag{11.10}$$

We wish to show that this tensor obeys the same transformation law (11.4) as the derivative D_{μ}. Indeed,

$$G_{\mu\nu} = D_{\mu}W_{\nu} - D_{\nu}W_{\mu} \to \omega D_{\mu}\omega^{-1}\left(\omega W_{\nu}\omega^{-1} + \frac{i}{e}\,\omega\partial_{\nu}\omega^{-1}\right)$$

$$-\omega D_{\nu}\omega^{-1}\left(\omega W_{\mu}\omega^{-1} + \frac{i}{e}\,\omega\partial_{\mu}\omega^{-1}\right) = \omega\,(D_{\mu}W_{\nu})\,\omega^{-1} - \omega\,(D_{\nu}W_{\mu})\,\omega^{-1}$$

$$+[\omega W_{\nu}\partial_{\mu}\omega^{-1} - \omega W_{\mu}\partial_{\nu}\omega^{-1}] + \frac{i}{e}\,[\omega D_{\mu}\partial_{\nu}\omega^{-1} - \omega D_{\nu}\partial_{\mu}\omega^{-1}].$$

Here in the last part of the expression the operators D_{μ} and D_{ν} within parentheses act only on W_{ν} and W_{μ}. The differentials of ω^{-1} are collected in the next term, which cancels the last term. Thus,

$$G_{\mu\nu} \to G'_{\mu\nu} = \omega G_{\mu\nu}\omega^{-1}. \tag{11.11}$$

11.2 Yang-Mills Fields

Let us consider the three-dimensional rotation group in the isotopic space. For the generators we have the commutation relation

$$[L_{a}, L_{b}] = i\varepsilon_{abc}L_{c} \quad (a, b, c = 1, 2, 3), \tag{11.12}$$

that is, the structure constants c_{abc} coincide with ε_{abc}. Then the infinitesimal transformation (11.9) consists of a rotation and a gauge transformation:

$$W'^{c}_{\mu} = W^{c}_{\mu} + \varepsilon_{abc}\theta^{a}W^{b}_{\mu} - \frac{1}{e}\partial_{\mu}\theta^{c}, \tag{11.13}$$

or, in vector form,

$$\mathbf{W}'_{\mu} = \mathbf{W}_{\mu} + \boldsymbol{\theta} \times \mathbf{W}_{\mu} - \frac{1}{e}\partial_{\mu}\boldsymbol{\theta}. \tag{11.14}$$

The components of the field-strength tensor (11.10) are

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\varepsilon_{abc} W^b_\mu W^c_\nu, \tag{11.15}$$

or

$$\mathbf{G}_{\mu\nu} = \partial_\mu \mathbf{W}_\mu - \partial_\mu \mathbf{W}_\mu + e\mathbf{W}_\mu \times \mathbf{W}_\nu. \tag{11.16}$$

The transformations of the components of the field-strength tensor are, in contrast to (11.14), pure rotations:

$$\delta \mathbf{G}_{\mu\nu} = \mathbf{G}'_{\mu\nu} - \mathbf{G}_{\mu\nu} = \boldsymbol{\theta} \times \mathbf{G}_{\mu\nu}. \tag{11.17}$$

Thus, for a gauge field (in Yang-Mills fields) we can introduce a Lagrangian density

$$\mathscr{L}_{\text{Y-M}} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \tag{11.18}$$

that is invariant under rotations belonging to the local $SU(2)$ group. In the spinor representation, the $\mathbf{L}_a$ are equal to $(1/2)\,\sigma_a$, where σ_a are the Pauli matrices. We get

$$W^\mu_a = \text{Tr}\ \mathbf{W}^\mu \sigma_a, \tag{11.19}$$

where Tr stands for "trace" of the matrix and where we have used the following property of the Pauli matrices:

$$\text{Tr}\ \sigma_a \sigma_b = 2\delta_{ab}. \tag{11.20}$$

Bearing this in mind, we can write (11.18) in the form

$$\mathscr{L}_{\text{Y-M}} = -\frac{1}{2} \text{Tr}\, \mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu}. \tag{11.21}$$

Let us now find the Yang-Mills field equations:

$$\frac{\partial \mathscr{L}}{\partial W^\mu_a} - \frac{\partial}{\partial x^\nu} \frac{\partial \mathscr{L}}{\partial W^\mu_{a,\,\nu}} = 0.$$

Employing (11.18), we get

$$\partial^\nu G^a_{\mu\nu} = e\varepsilon_{abc} G^b_{\mu\nu} W^\nu_c, \tag{11.22}$$

or, in compact form,

$$\mathbf{D}^\nu G^a_{\mu\nu} = 0, \tag{11.23}$$

where

$$\mathbf{D}^\nu G^a_{\mu\nu} = \partial^\nu G^a_{\mu\nu} + e\varepsilon_{abc} W^\nu_b G^c_{\mu\nu}.$$

If the gauge field is not free but interacts with a charged field, we must introduce in the right-hand side of Eq. (11.23) the density of the 4-current of the charged field:

$$\mathbf{D}^\nu G^a_{\mu\nu} = -j^a_\mu. \tag{11.23a}$$

We introduce the dual field-strength tensor

$$\tilde{G}_{\mu\nu} = \frac{1}{2} e_{\mu\nu\lambda\rho} G^{\lambda\rho}. \tag{11.24}$$

By the very definition of the field-strength tensor $G_{\mu\nu}$, (11.16), the dual tensor (11.24) satisfies the field equations identically:

$$D^{\nu}\tilde{G}^{a}_{\mu\nu} = 0. \tag{11.25}$$

The reader is advised to demonstrate this fact by direct calculation.

Equations (11.23) and (11.25) are similar to the Maxwell equations for a free electromagnetic field. However, there is an important difference here. First and foremost, the Yang-Mills field equations are nonlinear, which suggests that a Yang-Mills field is self-acting, that is, the various isotopic components of the field interact with each other. In addition, being invariant, the Yang-Mills equations nevertheless (in contrast to the Maxwell equations) along with the field-strength tensor contain the potentials W^{a}_{μ}, which in this way acquire a definite physical meaning. Consequently, the choice of a definite gauge (within which gauge the solution of the Yang-Mills equations is sought) also has a very definite physical meaning.

Because the Yang-Mills equations are nonlinear, it is extremely difficult to find exact solutions to these equations. Only some partial static solutions are known for a system of interacting Yang-Mills fields and charged scalar fields, solutions with a finite energy, the so-called monopoles. Also dynamical solutions in the Euclidean metric (with the imaginary time variable) are known. Such localized nonsingular solutions are known as instantons. Today instantons and monopoles occupy the center of the stage in theoretical physics. New solutions could shed light on the problem of nonlinear quantization of gauge fields.

11.3 The Higgs Mechanism in Classical Field Theory

Let us consider a model suggested by P. Higgs. The system consists of interacting Yang-Mills fields and a scalar isotriplet with self-action:

$$\mathcal{L} = -\frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu}_{a} + \frac{1}{2} (D_{\mu}\varphi^{a})(D^{\mu}\varphi_{a}) - V(\varphi). \tag{11.26}$$

The potential energy of self-action,

$$V(\varphi) = \frac{\lambda}{4}\left(\frac{m^2}{\lambda} - \varphi^2\right)^2, \quad \lambda > 0, \tag{11.27}$$

depends on the square $(\varphi)^2$ of the isovector field

$$\varphi^{a} = (\varphi^1, \varphi^2, \varphi^3), \tag{11.28}$$

and the derivative of φ^a has the form

$$D_\mu \varphi^a = \partial_\mu \varphi^a + e\varepsilon_{abc} W_\mu^b \varphi^c. \tag{11.29}$$

The last equation follows from definition (11.5),

$$(D_\mu \varphi)^a = [\delta_{ac}\partial_\mu - ie\,(L_b W_\mu^b)_{ac}]\,\varphi^c,$$

where in the vector representation, according to (3.66), the matrix of generator L_b is

$$(L_b)_{ac} = i\varepsilon_{abc}.$$

The Lagrangian is clearly gauge invariant under isotopic rotations belonging to the SU (2) group. Depending on the sign of m^2, the potential energy $V(\varphi)$ has a minimum either at $\varphi^2 = 0$ if m^2 is negative or at $\varphi^2 = m^2/\lambda$ if m^2 is positive. The classical vacuum is defined as the asymptotic form of the solution as $r \to \infty$ that ensures the minimum of the field energy. For m^2 negative, the vacuum proves to be symmetric,

$$\varphi \to 0 \quad \text{as } r \to \infty, \tag{11.30}$$

since it does not break the symmetry of the Lagrangian, while for m^2 positive the vacuum is asymmetric,

$$\varphi^a \to n^a \sqrt{m^2/\lambda} \quad \text{as } r \to \infty, \tag{11.31}$$

where n^a is an arbitrary unit vector, $|\mathbf{n}| = 1$, in the isospace. Obviously, the arbitrariness of n^a breaks the initial symmetry of SU (2) inherent in the Lagrangian and lowers the symmetry of the solution built by using this asymptotic behavior to the one-parameter group U (1).

Let us consider the more simple Abelian case. We assume that only two-dimensional rotations are possible about the third isotopic axis, rotations that form the U (1) group, which is a subgroup of SU (2). Then the potential W_μ will have only one nonzero component

$$W_\mu^a = \delta_{a3} W_\mu^3. \tag{11.32}$$

The result is a theory that is similar to electrodynamics with potential $A_\mu = W_\mu^3$. We go over to cylindrical coordinates ρ, θ, and φ^3, that is,

$$\varphi^1 = \rho \cos\theta, \quad \varphi^2 = \rho \sin\theta, \quad \varphi^3. \tag{11.33}$$

According to (11.29), for the derivatives we have

$$\begin{aligned} D_\mu \varphi^1 &= \cos\theta \partial_\mu \rho - e\rho \sin\theta \left(A_\mu + \frac{1}{e}\partial_\mu \theta\right), \\ D_\mu \varphi^2 &= \sin\theta \partial_\mu \rho + e\rho \cos\theta \left(A_\mu + \frac{1}{e}\partial_\mu \theta\right), \end{aligned} \tag{11.34}$$

and the Lagrangian (11.26) takes the form

$$\mathcal{L} = -\frac{1}{4} G^3_{\mu\nu} G_3^{\mu\nu} + \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} e^2 \rho^2 \left(A_\mu + \frac{1}{e} \partial_\mu \theta \right)^2 + \frac{1}{2} (\partial_\mu \varphi^3)^2 - \frac{\lambda}{4} \left(\frac{m^2}{\lambda} - \rho^2 - \varphi_3^2 \right)^2. \tag{11.35}$$

This Lagrangian is invariant with respect to the local gauge group $U(1)$. We perform the local gauge transformation of fields φ^1, φ^2, and A_μ corresponding to the rotation about axis *3* through an angle θ

$$\begin{gathered} \varphi^1 \to \varphi'^1 = \rho, \quad \varphi^2 \to \varphi'^2 = 0, \\ A_\mu \to A'_\mu = A_\mu + \frac{1}{e} \partial_\mu \theta, \quad G'^3_{\mu\nu} = G^3_{\mu\nu}. \end{gathered} \tag{11.36}$$

In this case the variable θ disappears from the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} (G^3_{\mu\nu})^2 + \frac{1}{2} (\partial_\mu \rho)^2 + \frac{1}{2} e^2 \rho^2 A'^2_\mu + \frac{1}{2} (\partial_\mu \varphi_3)^2 - \frac{\lambda}{4} \left(\frac{m^2}{\lambda} - \rho^2 - \varphi_3^2 \right)^2. \tag{11.37}$$

The self-action potential energy has a minimum at

$$\rho^2 + \varphi_3^2 = m^2/\lambda. \tag{11.38}$$

Let us take the vacuum values of the variables ρ and φ_3 satisfying condition (11.38):

$$\rho = \sqrt{m^2/\lambda} = \eta, \quad \varphi_3 = 0. \tag{11.39}$$

We introduce a new variable, $\rho' = \rho - \eta$, and write the Lagrangian near the vacuum (11.39) in the quadratic approximation:

$$\mathcal{L} = -\frac{1}{4} (G^3_{\mu\nu})^2 + \frac{1}{2} (\partial_\mu \rho')^2 + \frac{1}{2} e^2 \eta^2 A'^2_\mu + \frac{1}{2} (\partial_\mu \varphi_3)^2 - m^2 \rho'^2. \tag{11.40}$$

This Lagrangian describes three noninteracting fields: the vector field A'_μ and two scalar fields, ρ' and φ_3. Note that, while in the initial Lagrangian (11.26) the vector field had no mass term proportional to A^2_μ, in (11.40) such a term appears. The field A'_μ has acquired a mass, as follows from the comparison of (11.40) with the vector field Lagrangian (9.24), equal to

$$m_A = |e| \eta. \tag{11.41}$$

In this way the field A'_μ ceases to be purely transverse (as A_μ is) and acquires a third, longitudinal, polarization. The scalar field ρ' also proved to be massive, with a mass equal to

$$m_\rho = \sqrt{2}\, m, \tag{11.42}$$

while the field φ_3 remained massless.

Initially, the system had five degrees of freedom: the transverse field A_μ with two polarization states and the scalar Higgs triplet φ^a ($a = 1, 2, 3$). By appropriate selection of the gauge (11.36), one of the scalar Higgs fields, θ, disappeared from the Lagrangian, passing its degree of freedom to the vector field, which ceased to be transverse. Fixing the gauge (11.36) and choosing a definite vacuum state (11.39) about which the expansion that leads to Lagrangian (11.40) is performed resulted in the initial symmetry of the system being broken or, as it is usually said, in spontaneous symmetry breaking. The vector field acquires mass and the third polarization state. Of the two remaining Higgs fields one, ρ', also becomes massive. The number of degrees of freedom in the process, obviously, remains the same, five. Thus, as a result of spontaneous breaking of the local one-parameter symmetry, one massless scalar particle (a Goldstone boson; see Section 7) disappears, but the vector gauge field becomes massive. This is the essence of the Higgs mechanism of formation of mass of elementary particles.

If from the start we do not restrict our discussion to the one-parameter subgroup $U(1)$ but instead consider the total symmetry group for the Lagrangian (11.25), $SU(2)$, the vector field will have three isotopic components, W^a_μ ($a = 1, 2, 3$). In this case it can be demonstrated that the spontaneous symmetry breaking occurring because of the choice of the vacuum (11.31) lowers the symmetry from $SU(2)$ to $U(1)$ and diminishes the number of parameters of the group from three to one. Two of the three components of the vector field acquire mass, two Goldstone bosons vanish, and the third scalar Higgs particle becomes massive. A similar phenomenon related to the Higgs mechanism is observed in quantum theory. There it lies at the base of all unified theories of elementary particle interactions (the Weinberg-Salam-Glashow model for uniting electromagnetic and weak interactions and the grand unification theory, that is, the theory that unites the strong interaction with the electroweak).

Chapter II

The Basics of Classical Electrodynamics

12. The Maxwell-Lorentz Equations

The general theory of the electromagnetic field as a massless vector field was discussed in Section 10. The field is described by an antisymmetric second-rank tensor $F^{\mu\nu}$ (the field-strength tensor) that satisfies the system of Maxwell equation (see (10.18) and (10.36)):

$$\partial_\nu F^{\mu\nu}(x) = -4\pi j^\mu(x), \tag{12.1}$$

$$\partial_\nu \tilde{F}^{\mu\nu}(x) = 0, \tag{12.2}$$

where $\tilde{F}^{\mu\nu} = (1/2)\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the dual field-strength tensor (10.37), and $j^\mu = (\rho, \mathbf{j})$ is the 4-vector of the electromagnetic-current density consisting of the charge density ρ and current density $\mathbf{j}$.*

Let us write the Maxwell equations in three-dimensional vector form. To this end we introduce the electric-field strength $\mathbf{E}$ and the magnetic-field strength $\mathbf{H}$ according to (10.11a):

$$\operatorname{curl}\mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{j}, \quad \operatorname{div}\mathbf{E} = 4\pi\rho, \tag{12.1a}$$

$$\operatorname{curl}\mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} = 0, \quad \operatorname{div}\mathbf{H} = 0, \tag{12.1b}$$

where the first group of equations, (12.1a), follows from (12.1) and the second group, (12.1b), from (12.2). Note that the 3-vectors $\mathbf{E}$ and $\mathbf{H}$ are not the spatial components of any 4-vectors. Under Lorentz transformations, $\mathbf{E}$ and $\mathbf{H}$ are expressed in terms of each other and together form the field-strength tensor $F^{\mu\nu}$.

In classical electrodynamics the current j^μ is created by charged particles, so that

$$\rho(t, \mathbf{r}) = \sum_a e_a \delta(\mathbf{r} - \mathbf{r}_a(t)), \quad \mathbf{j}(t, \mathbf{r}) = \sum_a e_a \mathbf{v}_a \delta(\mathbf{r} - \mathbf{r}_a(t)), \tag{12.3}$$

* In contrast to Section 10, here the charge e is incorporated in the definition of the current j^μ (see formula (12.4)).

where e_a is the charge of the ath particle, $\mathbf{r}_a(t)$ and $\mathbf{v}_a = d\mathbf{r}_a/dt$ are the radius vector and the velocity of the ath particle, and $\delta(\mathbf{r} - \mathbf{r}_a)$ is the three-dimensional delta function.

For a single particle we can write an expression for the current that the particle creates in an explicitly covariant form:

$$j^\mu(x) = e\int_{-\infty}^{\infty} d\tau u^\mu(\tau)\,\delta_4(x - z(\tau)). \tag{12.4}$$

Here $z^\mu(\tau) = (t', \mathbf{r}_e(t'))$ is the four-dimensional radius vector of the particle, τ the proper time of the particle related to the laboratory time $t' = z^0$ through the well-known relationship

$$d\tau = (dz^2)^{1/2} = (1 - v^2)^{1/2}\,dt', \tag{12.5}$$

$$u^\mu = \frac{dz^\mu}{d\tau} = (1 - v^2)^{-1/2}(1, \mathbf{v}) \tag{12.6}$$

is the four-dimensional velocity (with $u^2 = u^\mu u_\mu = 1$), $\mathbf{v} = d\mathbf{r}_e/dt'$ is the common three-dimensional velocity, and δ_4 is the four-dimensional delta function

$$\delta_4(x - z(\tau)) = \delta(t - z^0(\tau))\,\delta(\mathbf{r} - \mathbf{r}_e(\tau)). \tag{12.7}$$

Integrating in (12.4) with respect to proper time τ via the formula for the delta function of a composite argument,

$$\delta[\varphi(\tau)] = \sum_s \frac{\delta(\tau - \tau_s)}{|\varphi'(\tau_s)|}, \tag{12.8}$$

where τ_s are the simple zeros of the function $\varphi(\tau)$, we arrive at a well-known formula (see (12.3)):

$$j^\mu(t, \mathbf{r}) = e\delta(\mathbf{r} - \mathbf{r}_e(t))(1, \mathbf{v}). \tag{12.4a}$$

It is readily seen that the 4-current j^μ created by a single particle, (12.4), satisfies the continuity equation (10.18a):

$$\partial_\mu j^\mu = -e\int_{-\infty}^{\infty} dz^\mu \frac{\partial}{\partial z^\mu}\delta_4(x - z) = -e\int d\delta_4(x - z(\tau))$$

$$= -e[\delta(x - z(\tau))]_{\tau=-\infty}^{\tau=+\infty} = 0.$$

The motion of a particle with charge e and mass m is fixed by the equation of classical mechanics

$$m\frac{du^\mu}{d\tau} = eF^{\mu\nu}u_\nu, \tag{12.9}$$

where the force is determined by the electromagnetic field $F^{\mu\nu}(x)$ at the point $x = z(\tau)$ where the particle is at time τ. In three-dimensional form, Eq. (12.9) becomes (see (10.11a) and (12.6))

$$\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1-v^2}} = e[\mathbf{E}(t, \mathbf{r}_e) + \mathbf{v}\times\mathbf{H}(t, \mathbf{r}_e)], \tag{12.9a}$$

where the expression on the right-hand side is known as the Lorentz force.

Equations (12.1), (12.2), (12.4), and (12.9) (or (12.1a), (12.1b), (12.3), and (12.9a) in three-dimensional notation) form a closed system of equations of classical electrodynamics, often called the Maxwell-Lorentz equations.

13. The Green Function of the Wave Equation

Let us now integrate the Maxwell-Lorentz equations (12.1) and (12.2) for a given current j^μ. As already noted in Section 10, such a statement of the problem is necessarily approximate, but in many cases it is justified and corresponds to the motion of charges in an external electromagnetic field $F^{\mu\nu}_{\text{ext}}$ that is strong compared to the field $F^{\mu\nu}$ created by the charges proper (in other words, by the current j^μ). The current j^μ may be determined by solving the equation of motion (12.9) of the charge with a given field-strength tensor $F^{\mu\nu}_{\text{ext}}$.

In Section 10 it was shown that the field-strength tensor $F^{\mu\nu}$ can be written in terms of the 4-potential $A^\mu(x)$ as (see (10.11))

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{13.1}$$

This representation ensures that Eq. (12.2) for the dual field-strength tensor $\tilde{F}^{\mu\nu}$ is satisfied automatically (see (10.39) above). The potential A^μ satisfies Eq. (10.19). Employing the fact that the electromagnetic-field equations (12.1) (or (10.19)) are invariant under local gauge transformations (10.9), we restrict the potential A^μ by the Lorentz condition

$$\partial_\mu A^\mu = 0. \tag{13.2}$$

Then Eq. (10.19) is reduced to the inhomogeneous wave equation (d'Alembert's equation; see (10.22)):

$$\Box A^\mu(x) = 4\pi j^\mu(x). \tag{13.3}$$

Its general solution can be written thus:

$$A^\mu(x) = 4\pi\int D(x-x')\, j^\mu(x')\, d^4x' + \overset{0}{A}{}^\mu(x), \tag{13.4}$$

where $\overset{0}{A}{}^{\mu}(x)$ is an arbitrary solution of the homogeneous equation

$$\Box \overset{0}{A}{}^{\mu}(x) = 0. \tag{13.5}$$

The function $D(x - x')$ in (13.4) is the kernel of the integral operator $\Box^{-1}$, the inverse of $\Box$, and satisfies the inhomogeneous equation with a special right-hand side, a delta function:

$$\Box_x D(x - x') = \delta_4(x - x'). \tag{13.6}$$

The function D is known as the Green function of the wave equation and has a simple physical meaning: it describes the perturbation in the form of a wave generated by an instantaneously acting point source.

We will find the solution to Eq. (13.6) by expanding D and δ_4 in four-dimensional Fourier integrals:

$$D(x) = \int \frac{d^4k}{(2\pi)^4} \widetilde{D}(k) e^{-ikx}, \tag{13.7}$$

$$\delta_4(k) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}, \tag{13.8}$$

with $kx \equiv k^{\mu}x_{\mu} = k^0t - \mathbf{k}\cdot\mathbf{r}$. Substituting these expansions into (13.6), we find the Fourier transform $\widetilde{D}(k)$ from the simple equation

$$k^2\widetilde{D}(k) = -1,$$

whose general solution has the form

$$\widetilde{D} = -\frac{1}{k^2} + C(k)\,\delta(k^2),$$

where $C(k)$ is an arbitrary nonsingular function of k^{μ}. However, the second term can be discarded here because allowing for it adds to $D(x)$ given by (13.7) a term that satisfies the homogeneous equation, which amounts to redefining the arbitrary function $\overset{0}{A}{}_{\mu}(x)$ in (13.4). Thus, we put

$$\widetilde{D}(k) = -\frac{1}{k^2}, \tag{13.9}$$

with $k^2 = k_0^2 - \mathbf{k}^2$. As a result we find that

$$D(x - x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2}. \tag{13.10}$$

This expression provides a formal solution to Eq. (13.6), since we still have not defined the rule of the integration of the singularity of $\widetilde{D}(k)$ at $k_0 = \pm|\mathbf{k}|$. To fix this rule, we must impose certain

boundary conditions on D. Starting from the physical meaning of the Green function, we require that

$$D_{\text{ret}}(x - x') = 0 \text{ for } x^0 < x'^0. \tag{13.11}$$

This requirement defines what is known as the retarded Green function $D_{\text{ret}}(x - x')$, which satisfies the causality principle (a perturbation appears after the source generated it):

$$D_{\text{ret}}(x) = -\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \int_{C_r} \frac{dk_0}{2\pi} \frac{e^{-ik_0(t-t')}}{k_0^2 - |\mathbf{k}|^2}, \tag{13.12}$$

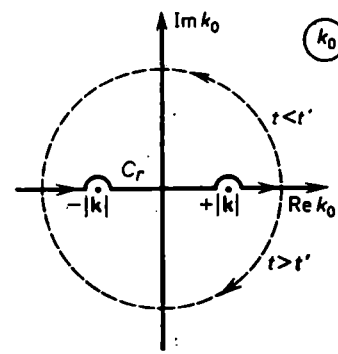

Figure 13.1

where the integration contour C_r in variable k_0 is shown in Figure 13.1. Employing Jordan's lemma known from the theory of functions of a complex variable, we can easily verify that for $t < t'$ the contour C_r must be closed in the upper half-plane, while for $t > t'$ it must be closed in the lower half-plane. Since the poles at $k_0 = \pm |\mathbf{k}|$ are by-passed from above, for $t < t'$ we get (13.11). For $t > t'$ the integral with respect to k_0 is reduced to the sum of residues at the poles:

$$\int_{C_r} \frac{dk_0}{2\pi} \frac{e^{-ik_0(t-t')}}{k_0^2 - \omega^2} = -i\left[\frac{e^{i\omega(t-t')}}{-2\omega} + \frac{e^{-i\omega(t-t')}}{2\omega}\right] = -\frac{\sin\omega(t-t')}{\omega},$$

where $\omega \equiv |\mathbf{k}|$, and we have allowed for the fact that the contour is traversed in the clockwise sense (hence the "minus"). Formula (13.12) becomes

$$D_{\text{ret}}(x - x') = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{R}} \frac{\sin\omega T}{\omega},$$

with $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $T = t - t'$.

Further integration is carried out in spherical coordinates, ω, θ, φ, in the $\mathbf{k}$-space: $d^3k = \omega^2 d\omega \sin\theta\, d\theta\, d\varphi$, and it is convenient to send the k_3 axis along vector $\mathbf{R}$, with $\mathbf{k}\cdot\mathbf{R} = \omega R\cos\theta$. Integration with respect to the angles yields

$$D_{\text{ret}}(x - x') = \frac{1}{2\pi^2 R} \int_0^\infty d\omega \sin\omega T \sin\omega R.$$

The remaining integral can be reduced to a delta function, $\delta(T-R)$, and we arrive at the retarded Green function in the form

$$D_{\text{ret}}(x-x') = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}\,\delta(t-t'-|\mathbf{r}-\mathbf{r}'|). \qquad (13.13)$$

This solution describes an infinitely narrow spherical wave that propagates with the speed of light and is generated by a source at point $\mathbf{r} = \mathbf{r}'$ acting instantaneously at time $t = t'$.

Employing the well-known formula

$$\delta(x^2-a^2) = \frac{1}{2|a|}\,[\delta(x-a)+\delta(x+a)], \qquad (13.14)$$

which follows from (12.8), we write D_{ret} in invariant form:

$$D_{\text{ret}}(x-x') = \frac{\theta(x_0-x_0')}{2\pi}\,\delta((x-x')^2). \qquad (13.15)$$

Here we have introduced the unit step function

$$\theta(z) = \begin{cases} 0, & z<0, \\ 1, & z>0. \end{cases} \qquad (13.16)$$

Note that the Lorentz invariance of the function D_{ret} (13.15) follows from the invariance of the sign of the temporal component $x_0 - x_0'$ of the 4-vector $x - x'$ under (proper) Lorentz transformations (see Section 3). Thus, $\theta(x^0)$ is a Lorentz-invariant function.

Let us now return to the representation (13.12) of the retarded Green function. We arrive at the same result when in integrating with respect to k_0 we move the poles at $k_0 = \pm|\mathbf{k}|$ downward into the lower half-plane by an infinitesimal: $k_0 = \pm|\mathbf{k}| - i\varepsilon$, with ε tending to $+0$, and send contour C_r along the real axis $\operatorname{Re} k_0$. This, obviously, is equivalent to replacing k_0 with $k_0 + i\varepsilon$ in (13.12). The result is an equivalent representation (which is used more often, we note)

$$D_{\text{ret}}(x-x') = -\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik(x-x')}}{k^2+i\varepsilon k_0}. \qquad (13.17)$$

The above replacement has a simple physical meaning: for a non-zero (but small) ε, Eq. (13.17) describes a damped wave:

$$D_{\text{ret}}(x) \propto \exp(-\varepsilon t).$$

Combining (13.4) with (13.15), we get the retarded solution of the wave equation:

$$A^{\mu}_{\text{ret}}(x) = 2\int \theta(x_0-x_0')\,\delta((x-x')^2)\,j^{\mu}(x')\,d^4x' + \overset{0}{A}{}^{\mu}(x), \qquad (13.18)$$

or in three-dimensional form (see (13.13)),

$$A^{\mu}_{\text{ret}}(t, \mathbf{r}) = \int \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} j^{\mu}(t', \mathbf{r}')\, dt'\, d^3x' + \overset{0}{A}{}^{\mu}(t, \mathbf{r}). \quad (13.18a)$$

Here $\overset{0}{A}{}^{\mu}(x)$ describes a field created by external sources that lie outside the system being studied. Indeed, if $j^{\mu}(t, \mathbf{r}) = 0$ for $t < t_0$, then (13.18) yields $A^{\mu}_{\text{ret}} = \overset{0}{A}{}^{\mu}$ for $t < t_0$.

We note in passing that the general solution of the homogeneous wave equation, $\overset{0}{A}{}^{\mu}(x)$, can be represented as an expansion in plane monochromatic waves (see also Section 10):

$$\overset{0}{A}{}^{\mu}(x) = \int \frac{d^4k}{(2\pi)^4} a^{\mu}(k)\, \delta(k^2)\, e^{-ikx}. \quad (13.19)$$

Here $a^{\mu}(k)$ is an arbitrary (nonsingular) 4-vector function of k that in the Lorentz gauge (13.2) satisfies the transversality condition:

$$ka = k^0a^0 - \mathbf{k}\cdot\mathbf{a} = 0. \quad (13.20)$$

The presence of the delta function $\delta(k^2)$ in (13.19) expresses the well-known relationship between the frequency of a monochromatic wave, $\omega = k_0$, and the wave vector $\mathbf{k}$: $k_0^2 - \mathbf{k}^2 = 0$.

It can also be easily verified that, as expected (see Section 10), the field strengths $\mathbf{E}$ and $\mathbf{H}$ in a plane monochromatic wave are orthogonal to each other and to vector $\mathbf{k}$. Indeed, according to (10.11a), for the components of the field strengths in such a wave we have

$$E_l = k^0a^l - k^la^0, \quad H_l = -\frac{1}{2}\varepsilon_{lmn}(k^ma^n - k^na^m).$$

Combining this with (13.20), we find that

$$\mathbf{k}\cdot\mathbf{E} = k^lE_l = k^0(\mathbf{a}\cdot\mathbf{k}) - \mathbf{k}^2a^0 = a^0(k_0^2 - \mathbf{k}^2) = 0,$$
$$\mathbf{k}\cdot\mathbf{H} = k^lH_l = 0, \quad \mathbf{E}\cdot\mathbf{H} = E_lH_l = 0,$$

since $k^lk^m\,\varepsilon_{lmn} \equiv 0$ and $a^la^m\,\varepsilon_{lmn} \equiv 0$, in view of the fact that ε_{lmn} is antisymmetric.

In what follows we consider only fields created by the current j^{μ} proper, so that in (13.18) we can ignore $\overset{0}{A}{}^{\mu}(x)$ (which represents a field that "impinges" on the system from outside).

Let us now construct the so-called advanced Green function D_{adv}, which satisfies the following condition:

$$D_{\text{adv}}(x - x') = 0, \text{ for } x^0 > x'^0. \quad (13.21)$$

This function, of course, has no direct physical meaning, but is used in the theory of radiation (see Section 16).

It can easily be verified that the Fourier expansion is obtained from (13.17) by replacing ε with $-\varepsilon$ (which corresponds to traversal of the poles in Figure 13.1 from below),

$$D_{\text{adv}}(x-x') = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - ik_0}, \tag{13.22}$$

or by reversing the sign of the argument in the unit step function in (13.15):

$$D_{\text{adv}}(x-x') = \frac{\theta(-(x_0-x_0'))}{2\pi} \delta((x-x')^2). \tag{13.23}$$

The difference between the retarded and advanced Green functions obeys a homogeneous wave equation. For future applications we write the Fourier expansion for the half-difference

$$D_{\text{rad}}(x) \equiv \frac{1}{2}[D_{\text{ret}}(x) - D_{\text{adv}}(x)] = \frac{1}{4\pi} \delta(x^2) \operatorname{sgn} x^0. \tag{13.24}$$

We use the well-known formula

$$\frac{1}{z \pm i0} = \mathcal{P}\frac{1}{z} \mp i\pi\delta(z), \tag{13.25}$$

which has the following meaning:

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} \frac{\varphi(z)}{z \pm i\varepsilon} dz = \mathcal{P} \int_{-\infty}^{\infty} \frac{\varphi(z)}{z} dz \mp i\pi\varphi(0), \tag{13.25a}$$

where $\mathcal{P}$ stands for the principal value of the integral, and $\varphi(z)$ is a well-behaved function that is regular at $z = 0$.

Combining (13.24), (13.17), (13.22), and (13.25), we arrive at the plane-wave expansion of D_{rad}:

$$D_{\text{rad}}(x) = 2\pi i \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \operatorname{sgn} k^0 \delta(k^2), \tag{13.26}$$

where we have introduced the sign function

$$\operatorname{sgn} z = \theta(z) - \theta(-z) = \frac{z}{|z|} = \begin{cases} -1, & z < 0, \\ +1, & z > 0. \end{cases} \tag{13.27}$$

14. The Variational Principle in Electrodynamics

The system of equations of electrodynamics, (12.1), (12.2), (12.9), may be obtained from the variational principle (see Section 1)

$$\delta S = 0. \tag{14.1}$$

Here S is the action integral of a system consisting of an electromagnetic field and charged particles interacting with this field. It

consists of three parts:

$$S = S_f + S_e^0 + S_{\text{int}}. \tag{14.2}$$

The first term represents the action integral for a field in the absence of charge (Section 10):

$$S_f = \int \mathcal{L}_f \, d^4x, \tag{14.3}$$

where $\mathcal{L}_f = -(1/16\pi)F^{\mu\nu}F_{\mu\nu} = (1/8\pi)(\mathbf{E}^2 - \mathbf{H}^2)$ is the Lagrangian density of the free field.

The second term represents the action integral for free particles. If, for the sake of simplicity, we take only one particle of mass m, then

$$S_e^0 = \int \mathcal{L}_e^0 \, ds, \tag{14.4}$$

where $\mathcal{L}_e^0 = -m(\dot{z}_\mu \dot{z}^\mu)^{1/2}$ is the Lagrangian density of the free particle, $\dot{z}^\mu = dz^\mu/ds$, and $z^\mu = z^\mu(s)$ is the particle's four-dimensional radius vector determined by the arbitrary parameter s that can be identified, after variation of the action integral has been carried out, with the proper time of the particle. For a system of particles, S_e^0 is the sum of type (14.4) terms over all the particles.

The third term in the action integral represents the interaction of the particles with the electromagnetic field:

$$S_{\text{int}} = \int \mathcal{L}_{\text{int}} \, d^4x, \quad \mathcal{L}_{\text{int}} = -A^\mu(x) \, j_\mu(x), \tag{14.5}$$

and for one particle we have

$$S_{\text{int}} = -e \int A^\mu(x) \, \delta_4 \, (x - z(s)) \, \dot{z}_\mu \, d^4x \, ds = -e \int A^\mu(z(s)) \, \dot{z}_\mu(s) \, ds, \tag{14.5a}$$

where we have allowed for (12.4).

Note (see Section 10) that within this framework the electromagnetic field is described by a 4-potential A^μ, whose components are the independent generalized coordinates subject to variation. The antisymmetric field-strength tensor (13.1), for this reason, satisfies Eq. (12.2) automatically. Equation (12.1) is obtained by varying (the current j^μ remains fixed in the process) the sum of two terms, S_f (14.3) and S_{int} (14.5), in A^μ. The necessary calculations were carried out in Section 10 (see (10.17) and (10.18)).

The equation of motion of a charged particle in an electromagnetic field follows from the action integral (see (14.4) and (14.5))

$$S_e = S_e^0 + S_{\text{int}} = \int L_e \, ds, \tag{14.6}$$

where $L_e = -m(\dot{z}_\nu \dot{z}^\nu)^{1/2} - eA^\nu(z)\dot{z}_\nu$ is the Lagrangian of the particle interacting with the field.

Variation of the action integral (14.6) in z^μ (for a fixed field A^μ) results in the common Lagrange equation

$$\frac{d}{ds}\frac{\partial L_e}{\partial \dot{z}_\mu} - \frac{\partial L_e}{\partial z_\mu} = 0. \tag{14.7}$$

Here

$$\partial L_e/\partial z_\mu = -e\dot{z}_\nu \partial A^\nu/\partial z_\mu \equiv -e\dot{z}_\nu \partial^\mu A^\nu(z),$$

$$\partial L_e/\partial \dot{z}_\mu = -m(\dot{z}^2)^{-1/2}\dot{z}^\mu - eA^\mu(z),$$

$$\frac{d}{ds}\frac{\partial L_e}{\partial \dot{z}_\mu} = -m\frac{d}{ds}\left(\frac{\dot{z}^\mu}{\sqrt{\dot{z}^2}}\right) - e\dot{z}_\nu \partial^\nu A^\mu.$$

Substituting these expressions into (14.7), we arrive at an equation of motion for a charged particle in the following form:

$$m\frac{d}{ds}\left(\frac{\dot{z}^\mu}{\sqrt{\dot{z}^2}}\right) = eF^{\mu\nu}(z)\dot{z}_\nu. \tag{14.8}$$

Let us now select the proper time of the particle, τ, as the parameter s (which up till now was arbitrary; see (12.5)). Then $\dot{z}^2 = 1$, and Eq. (14.8) transforms into the known equation (12.9). But if we put $s = t = z^0$ (the laboratory time), then $\dot{z}^\mu = (1, \mathbf{v})$, where $\mathbf{v} = d\mathbf{r}_e/dt$ is the common velocity. As a result, for $\mu = 1, 2, 3$ we get the equation of motion in the three-dimensional form (12.9a), while the temporal component ($\mu = 0$) in (14.8) yields the law of variation of kinetic energy:

$$\frac{d}{dt}\frac{m}{\sqrt{1-v^2}} = e\mathbf{v}\cdot\mathbf{E}. \tag{14.9}$$

15. Conservation of Energy-Momentum in Electrodynamics

We will now show that the energy and momentum of a system consisting of a field and particles are conserved, both quantities defined as the corresponding sums of energies and momenta of the electromagnetic field and the charged particles.

We start with the equation of motion of a particle in four-dimensional form, (12.9). Multiplying both sides by the delta function $\delta_4(x - z(\tau))$, where $z^\mu(\tau)$ is the four-dimensional radius vector of

the particle, integrating with respect to the proper time τ, and allowing for the expression (12.4) for the current density j^μ, we get

$$m \int_{-\infty}^{\infty} d\tau \frac{du^\mu}{d\tau} \delta_4 (x - z(\tau)) = F^{\mu\nu}(x) j_\nu (x).$$

Integrating the left-hand side of this relationship by parts and allowing for the fact that $d\delta_4 (x - z(\tau)) = - u^\nu (\tau) [\partial \delta_4 (x - z)/\partial x^\nu] d\tau$, we find that

$$\partial_\nu \overset{e}{T}{}^{\mu\nu}(x) = F^{\mu\nu}(x) j_\nu (x). \tag{15.1}$$

Here we have introduced the tensor

$$\overset{e}{T}{}^{\mu\nu}(x) = m \int_{-\infty}^{\infty} d\tau u^\mu u^\nu \delta_4 (x - z(\tau)), \tag{15.2}$$

which, in accord with the general definition (see Section 4), can be taken as the energy-momentum tensor of a system consisting of a single (point-like) particle. Indeed, integrating in (15.2) with respect to τ, we get

$$\overset{e}{T}{}^{\mu\nu} = m\delta (\mathbf{r} - \mathbf{r}_e (t)) u^\mu u^\nu \frac{d\tau}{dt}, \tag{15.2a}$$

which implies that

$$\overset{e}{T}{}^{00} = m\delta (\mathbf{r} - \mathbf{r}_e (t)) u^0 \tag{15.2b}$$

is the energy density, $\overset{e}{T}{}^{0k} = \overset{e}{T}{}^{00} u^k$ is the momentum density (the energy flux), and $\overset{e}{T}{}^{ik}$ forms the three-dimensional momentum-flux-density tensor (precisely, $\overset{e}{T}{}^{ik}$ is the ith component of the momentum vector flowing through a unit area that lies perpendicularly to the x^k axis).

For a system of many particles, the energy-momentum tensor is represented, obviously, by a sum of (15.2) type summands, or

$$\overset{e}{T}{}^{\mu\nu}(x) = \sum_a m_a \int_{-\infty}^{\infty} d\tau \delta_4 (x - z_a (\tau)) u_a^\mu u_a^\nu. \tag{15.2c}$$

Equation (15.1) remains valid in this case, too.

Now let us transform the right-hand side of (15.1), which represents the density of the force with which the field acts on the system of charges:

$$f^{\mu}(x) = F^{\mu\nu}(x)\, j_{\nu}(x). \tag{15.3}$$

Let us demonstrate that this quantity can be written in the form of a four-dimensional divergence:

$$f^{\mu} = -\partial_{\mu} \overset{f}{T}{}^{\mu\nu}, \tag{15.4}$$

where

$$\overset{f}{T}{}^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}_{\alpha} + \frac{1}{4}\, g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{15.5}$$

was introduced in Section 10 as the energy-momentum tensor of an electromagnetic field.

Indeed, substituting into (15.3) the current j_{ν} expressed in terms of the field-strength tensor through the Maxwell-Lorentz equation (12.1), we get

$$4\pi f^{\mu} = -F^{\mu\alpha}\partial^{\nu}F_{\alpha\nu} = -\partial^{\nu}(F^{\mu\alpha}F_{\alpha\nu}) + F_{\alpha\nu}(\partial^{\nu}F^{\mu\alpha}). \tag{15.6}$$

Let us now write the field equation (12.2) as follows:

$$\partial^{\nu}F^{\mu\alpha} + \partial^{\alpha}F^{\nu\mu} + \partial^{\mu}F^{\alpha\nu} = 0, \tag{15.7}$$

where we have allowed for the fact that $F^{\mu\nu}$ is antisymmetric. Multiplying this equations by $F_{\alpha\nu}$ and summing over ν, we find that

$$F_{\alpha\nu}\partial^{\nu}F^{\mu\alpha} = -\frac{1}{2}F_{\alpha\nu}\partial^{\mu}F^{\alpha\nu} = -\frac{1}{4}\,\partial^{\mu}(F_{\alpha\beta}F^{\alpha\beta}). \tag{15.8}$$

From (15.6) and (15.8) we obtain

$$4\pi f^{\mu} = -\partial^{\nu}\left[F^{\mu\alpha}F_{\alpha\nu} + \frac{1}{4}\,\delta^{\mu}_{\nu}F_{\alpha\beta}F^{\alpha\beta}\right],$$

which is actually (15.4).

Combining (15.1), (15.3), and (15.4), we arrive at the sought-for law of energy-momentum conservation in the form

$$\partial_{\nu}(\overset{e}{T}{}^{\mu\nu} + \overset{f}{T}{}^{\mu\nu}) = 0, \tag{15.9}$$

or, in view of (15.1),

$$\partial_{\nu}\overset{f}{T}{}^{\mu\nu} = -F^{\mu\nu}j_{\nu}. \tag{15.9a}$$

For greater clarity we write the conservation law in integral form.* Denoting the total energy-momentum tensor by $T^{\mu\nu}$ and employing (15.9), we get

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t}+\frac{\partial T^{0k}}{\partial x^k}=0.$$

Integrating this equation over a certain volume of three-dimensional space, V, bounded by surface σ and employing the divergence theorem, we get

$$\frac{d}{dt}\int_V T^{00}\,d^3x=-\oint_\sigma T^{0k}\,d\sigma_k, \tag{15.10}$$

with $d\sigma_k$ the components of the 3-vector of a surface element, $d\sigma = \mathbf{n}d\sigma$ ($\mathbf{n}$ is the unit outward vector normal to the surface). According to (15.10), the rate with which the energy of the system inside V varies is equal to the energy flux across the boundary surface σ.

Using (15.2c) and (15.5), we can rewrite (15.10) in a more pictorial form:

$$\frac{d}{dt}\left\{\int_V w d^3x+\sum_a \varepsilon_a\right\}=-\oint_\sigma \mathbf{S}\,d\sigma, \tag{15.11}$$

where

$$w=\frac{1}{8\pi}(\mathbf{E}^2+\mathbf{H}^2) \tag{15.12}$$

is the electromagnetic-field energy density ($w = \overset{f}{T}{}^{00}$),

$$\mathbf{S}=\frac{c}{4\pi}[\mathbf{E}\times\mathbf{H}] \tag{15.13}$$

is the flux-density vector of the electromagnetic-field energy ($S^k = c\overset{f}{T}{}^{0k}$), known as the Poynting vector, and $\varepsilon_a = m_a c^2 u_a^0 = m_a c^2(1-\mathrm{v}_a^2/c^2)^{-1/2}$ is the kinetic energy of the ath particle.

Similarly, we can arrive at the integral analog of (15.9a) in three-dimensional form:**

$$\frac{d}{dt}\int_V w\,d^3x=-\int_V \mathbf{j}\cdot\mathbf{E}\,d^3x-\oint_\sigma \mathbf{S}d\sigma, \tag{15.14}$$

* We use an ordinary system of units up to the end of this section.

** Note that both (15.11) and (15.14) have been written on the assumption that there are no particles on σ at a given moment (otherwise one must add to the right-hand sides of these equations a term that represents the energy flux of the particles that lie on σ, that is, $-\int_\sigma \overset{e}{T}{}^{0k}d\sigma_k$).

that is, the rate with which the energy of the field inside volume V varies is equal to the sum (with the "minus" sign) of the work performed (per unit time) by the field on the charges inside volume V and the field-energy flux across the boundary σ. Equation (15.14) can also be obtained directly from (15.11) if we allow for the law of variation (14.9) of the kinetic energy of a charged particle.

In the same manner, if we consider the components $T^{i\mu}$ of the energy-momentum tensor, we can obtain the law of conservation of the system's momentum (cf. (15.11)):

$$\frac{d}{dt}\left\{\int_V \mathbf{P}\,d^3x + \sum_a \mathbf{p}_a\right\} = -\oint_\sigma \mathbf{T}\,d\sigma, \tag{15.11a}$$

where

$$\mathbf{P} = \mathbf{S}/c^2 = \frac{1}{4\pi c}[\mathbf{E}\times\mathbf{H}] \tag{15.13a}$$

is the electromagnetic-field momentum density, $\mathbf{T}$ is a vector with components $T^i = T^{ik}n_k$ (the momentum-flux-density vector), and $\mathbf{p}_a = m_a\mathbf{v}_a\,(1 - \mathbf{v}_a^2/c^2)^{-1/2}$ is the momentum of the ath particle.

Allowing for the law of variation of particle momentum, we can rewrite Eq. (15.11a) as follows (cf. (15.14)):

$$\frac{d}{dt}\int_V \mathbf{P}\,d^3x = -\int_V \left(\rho\mathbf{E} + \frac{1}{c}\,\mathbf{j}\times\mathbf{H}\right) d^3x - \oint_\sigma \mathbf{T}\,d\sigma. \tag{15.14a}$$

According to (15.14a), the rate with which the electromagnetic-field momentum varies inside volume V is equal to the sum (taken with the "minus" sign) of the total Lorentz force that acts on the charges in this volume and the momentum flux across the boundary surface σ.

16. The Classical Theory of Radiation

We will now turn to a more detailed study of the field created by a point charge moving in an arbitrary manner. If the charge is being accelerated (or decelerated), its field consists of two quite distinct components. One does not depend on the acceleration and is simply the field of a uniformly moving charge. The other is determined by the charge's acceleration and at great distances R from the charge becomes predominant, since it decreases as the distance increases much slower ($\propto R^{-1}$) than the first component ($\propto R^{-2}$), which results in a nonzero electromagnetic-field energy flux, that is, radiation.

16.1 The Field of a Point Charge in Arbitrary Motion

Suppose that a charged particle is moving according to a given law: $z^\mu = z^\mu(\tau)$. The reader will recall that z^μ is the four-dimensional radius vector of the particle and τ is the particle's proper time (see (12.5)). We will find the field generated by the moving particle by using the general solution to the wave equation (see Section 13). Substituting the explicit form of the current, (12.4), into (13.18) and integrating with respect to x' via the delta function $\delta_4(x' - z)$, we obtain the retarded 4-potential of the field of the charge (the solution of the homogeneous equation, $\overset{0}{A}{}^\mu$, is discarded):

$$A^\mu_{\text{ret}}(x) = 2e \int_{-\infty}^{\infty} d\tau \theta(x^0 - z^0(\tau)) \delta((x - z(\tau))^2) u^\mu(\tau). \quad (16.1)$$

The remaining integration with respect to proper time is carried out by means of the relationship

$$\int_{-\infty}^{\infty} d\tau \theta(x^0 - z^0(\tau)) \delta((x - z(\tau))^2) \varphi(\tau) = \frac{\varphi(\tau_r)}{2u(\tau_r)(x - z(\tau_r))}, \quad (16.2)$$

which is a corollary of the well-known formula for the delta function of a composite argument, (12.8). Here τ_r is a root of the equation

$$(x - z(\tau_r))^2 = 0, \quad z^0(\tau_r) < t, \quad (16.3)$$

or, in three-dimensional form,

$$t - t_r = |\mathbf{r} - \mathbf{r}_e(t_r)|, \quad (16.3a)$$

where $t_r = z^0(\tau_r)$.

We can easily see that Eq. (16.3) has a unique solution. Indeed, let us consider the function $\varphi(\tau) = (x - z(\tau))^2$. Its derivative is

$$\frac{d\varphi}{d\tau} = -2u^\mu(x - z)_\mu = -2u^0(x^0 - z^0)[1 - \mathbf{v}\cdot\mathbf{R}/(x^0 - z^0)] < 0,$$

($\mathbf{R} = \mathbf{r} - \mathbf{r}_e$), since the velocity of the particle, v, is less than unity. Hence, $\varphi(\tau)$ is a monotone function of τ, and therefore τ_r is the only root of the equation.

Using (16.1), we can rewrite it in the form

$$A^\mu_{\text{ret}}(x) = \frac{eu^\mu}{(uX)}\Big|_{\tau=\tau_r}, \quad (16.4)$$

where we have introduced the notation

$$X^\mu = x^\mu - z^\mu(\tau), \quad X^2(\tau_r) = 0. \quad (16.5)$$

We write (16.4) in three-dimensional form:

$$\Phi_{\text{ret}}(t, \mathbf{r}) = \frac{e}{R - \mathbf{v}\cdot\mathbf{R}}\Big|_{t_r}, \quad \mathbf{A}_{\text{ret}}(t, \mathbf{r}) = \frac{e\mathbf{v}}{R - \mathbf{v}\cdot\mathbf{R}}\Big|_{t_r}, \quad (16.4a)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}_e$, $R = |\mathbf{R}|$, and t_r is the root of Eq. (16.3a). The potentials (16.4a) are known as the Liénard-Wiechert potentials. According to (16.4a) and (16.3a), the field generated by a moving charged particle and observed at point $\mathbf{r}$ in space and at t in time is determined by the position of the same particle at point $\mathbf{r}_e(t_r)$ at a previous point in time t_r, and $t - t_r$ is the time required for the electromagnetic perturbation to travel the distance $|\mathbf{r} - \mathbf{r}_e(t_r)|$.

Let us now calculate the field-strength tensor $F^{\mu\nu}_{\text{ret}}$, which is related to the potential (16.4) through (13.1). It is convenient to begin with the integral representation (16.1), which we will write in the form

$$A^\mu(x) = 2e\int_{-\infty}^{\infty} d\tau u^\mu(\tau)\,\theta(X^0)\,\delta(X^2), \quad (16.1a)$$

where for the sake of brevity we have dropped the subscript "ret". (Here we have used notation (16.5).) Whence*

$$\partial^\nu A^\mu = 2e\int_{-\infty}^{\infty} d\tau u^\mu \theta(X^0)\frac{\partial}{\partial x_\nu}\delta(X^2).$$

Let us transform the derivative of the delta function thus:

$$\frac{\partial}{\partial x_\nu}\delta(X^2) = 2X^\nu\frac{d\delta}{dX^2} = -\frac{X^\nu}{(uX)}\frac{d}{d\tau}\delta(X^2),$$

since $dX^2/d\tau = 2X^\mu(dX_\mu/d\tau) = -2(uX)$, in view of (16.5). Hence, $\partial^\nu A^\mu$ is reduced to

$$\partial^\nu A^\mu = -2e\int_{-\infty}^{\infty} d\tau\theta(X^0)\frac{u^\mu X^\nu}{(uX)}\frac{d}{d\tau}\delta(X^2),$$

or, after integration by parts (see the above footnote),

$$\partial^\nu A^\mu = 2e\int_{-\infty}^{\infty} d\tau\theta(X^0)\,\delta(X^2)\frac{d}{d\tau}\left[\frac{u^\mu X^\nu}{(uX)}\right]. \quad (16.6)$$

* It is easily verified that differentiation of the theta function contributes nothing to the result.

Calculating the integral (16.6) with due regard for (16.2), we find that

$$\partial^\nu A^\mu = \frac{e}{(uX)} \frac{d}{d\tau} \left[\frac{u^\mu X^\nu}{(uX)} \right] \Big|_{\tau_r}, \tag{16.7}$$

with the result that the field-strength tensor is

$$F^{\mu\nu}_{\text{ret}}(x) = \frac{e}{(uX)} \frac{d}{d\tau} \left[\frac{X^\mu u^\nu - X^\nu u^\mu}{(uX)} \right] \Big|_{\tau_r}. \tag{16.8}$$

By evaluating the derivatives we find the final expression for the electromagnetic field-strength tensor in the case of a field generated by a moving charge:

$$F^{\mu\nu}_{\text{ret}}(x) = \frac{e}{(uX)^2} \left\{ (X^\mu w^\nu - X^\nu w^\mu) - \frac{(wX)-1}{(uX)} (X^\mu u^\nu - X^\nu u^\mu) \right\} \Big|_{\tau_r}, \tag{16.9}$$

where we have introduced the four-dimensional acceleration

$$w^\mu = \frac{du^\mu}{d\tau} = \left\{ \frac{\mathbf{v}\cdot\mathbf{a}}{(1-v^2)^2}, \frac{\mathbf{a}}{1-v^2} + \frac{\mathbf{v}(\mathbf{v}\cdot\mathbf{a})}{(1-v^2)^2} \right\} \tag{16.10}$$

(with $\mathbf{a} = d\mathbf{v}/dt$ the three-dimensional acceleration) and allowed for the fact that $u^2 = u^\mu u_\mu = 1$. This also implies that w and u are orthogonal to each other, $(wu) = 0$.

Now we write the expressions for the electric and magnetic fields, which are related to the tensor (16.9) through (10.11a):

$$\mathbf{E} = e \frac{1-v^2}{R^2(1-\mathbf{n}\cdot\mathbf{v})^3} (\mathbf{n}-\mathbf{v}) + \frac{e}{R(1-\mathbf{n}\cdot\mathbf{v})^3} [\mathbf{n} \times ((\mathbf{n}-\mathbf{v}) \times \mathbf{a})],$$
$$\mathbf{H} = \mathbf{n} \times \mathbf{E}, \tag{16.11}$$

where $\mathbf{n} = \mathbf{R}/R$ is the unit vector, and all quantities on the right-hand sides of (16.11) are taken at time t_r (see (16.3a)). Note that vectors E and H are orthogonal to each other.

Let us now study (16.11) in greater detail. The electric field consists of two essentially different terms:

$$\mathbf{E} = \mathbf{E}^{(1)} + \mathbf{E}^{(2)}. \tag{16.11a}$$

The first term is independent of the particle acceleration and at large distances R diminishes like R^{-2}. It represents the field of a uniformly moving charge. Indeed, at $\mathbf{v} = \text{constant}$ we have

$$\mathbf{R}(t_r) - \mathbf{v}R(t_r) = \mathbf{R}(t_r) - \mathbf{v}(t - t_r) = \mathbf{R}(t) \equiv \mathbf{r} - \mathbf{r}_e(t),$$

the vector that links the point at which the charge is at the moment of observation with the observation point. This implies that

$$R(1-\mathbf{n}\cdot\mathbf{v})\big|_{t_r} = R\sqrt{1-v^2\sin^2\theta}\,\big|_t,$$

where θ is the angle between $\mathbf{R}(t)$ and $\mathbf{v}$. As a result, $\mathbf{E}^{(1)}$ is reduced to the well-known expression for the electric field of a uniformly moving charge:

$$\mathbf{E}^{(1)} = \frac{e\mathbf{R}}{R^3} \frac{1-v^2}{(1-v^2 \sin^2 \theta)^{3/2}}. \tag{16.12}$$

The second term in (16.11a), $\mathbf{E}^{(2)}$, depends on the acceleration $\mathbf{a}$ of the charge and diminishes like R^{-1} at distances (see (16.11))

$$R \gg (1 - v^2)/a, \tag{16.13}$$

where it becomes predominant and acquires the shape of a spherical wave: $|\mathbf{E}| = |\mathbf{H}| \propto r^{-1}$ (R is approximately equal to r if condition (16.13) is met), with the vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{n} = \mathbf{r}/r$ forming a right-handed triple of mutually perpendicular vectors, that is, the electromagnetic field of a charge that is being accelerated (or decelerated) is transverse far away from this charge. As we will now see, it is this term, which is nonzero only when $\mathbf{a} \neq 0$, that describes the emission of electromagnetic waves by a moving charge.

16.2 Power Radiated by an Accelerated Charge

Let us calculate the amount of energy radiated by a moving charge. The energy flux is given by the Poynting vector $\mathbf{S}$ (15.13). The electromagnetic-field energy $d\varepsilon$ emitted by a moving charge into a solid angle element $d\Omega$ in the direction of vector $\mathbf{n} = \mathbf{R}/R$ during the time interval from t_r to $t_r + dt_r$ will pass through the area element $d\sigma = R^2 d\Omega$ during the time interval from t to $t + dt$, with the moment of radiation t_r related to the observation moment t through (16.3a). This energy is equal to

$$d\varepsilon = (\mathbf{S} \cdot \mathbf{n}) R^2 d\Omega dt. \tag{16.14}$$

At large distances from the moving charge, in the so-called wave zone, where $|\mathbf{r}_e| \ll |\mathbf{r}|$ and condition (16.13) is met, we can put

$$\mathbf{S} = \frac{|\mathbf{E}|^2}{4\pi} \mathbf{n}. \tag{16.15}$$

Here $\mathbf{E}$ is determined by the second (wave) term in (16.11) (see Section 16.1), and within the same approximation we have $R \simeq r$ (it is assumed that the charge moves within a limited region containing the origin of the system of coordinates), $\mathbf{n} = \mathbf{r}/r$, while the relationship (16.3a) that links t_r and t simplifies:

$$t - t_r = [\mathbf{r}^2 - 2\mathbf{r} \cdot \mathbf{r}_e(t_r) + \mathbf{r}_e^2(t_r)]^{1/2} \simeq r \left(1 - 2 \frac{\mathbf{n} \cdot \mathbf{r}_e}{r}\right)^{1/2},$$

or

$$t = t_r + r - \mathbf{n} \cdot \mathbf{r}_e(t_r). \tag{16.16}$$

According to (16.14), (16.15), and (16.11), the radiation intensity, that is, the amount of energy registered by the observer per unit time t is

$$\frac{dI}{d\Omega} = \frac{d\mathcal{E}}{d\Omega\, dt} = \frac{e^2}{4\pi} \frac{[\mathbf{n} \times [(\mathbf{n}-\mathbf{v}) \times \mathbf{a}]]^2}{(1-\mathbf{n}\cdot\mathbf{v})^6}. \tag{16.17}$$

More exactly, (16.17) is known as the angular distribution of the radiation intensity, in contrast to the total intensity $I = d\mathcal{E}/dt$. We note once more that all the quantities on the right-hand side of (16.17) are taken at time t_r (see (16.16)).

More often, however, another quantity associated with radiation is of interest, that is, the energy emitted by the particle per unit time t_r, or the rate of loss of energy by the particle, $W = d\mathcal{E}/dt_r$. This quantity is known as the radiated power and differs from the radiation intensity. Indeed, from (16.16) we get

$$dt = dt_r\,(1 - \mathbf{n}\cdot\mathbf{v}), \tag{16.18}$$

from which follows the relationship that links the radiated power and the radiation intensity:

$$\frac{dW}{\partial\Omega} \equiv \frac{d\mathcal{E}}{dt_r\, d\Omega} = (1-\mathbf{n}\cdot\mathbf{v})\frac{dI}{d\Omega}. \tag{16.19}$$

From (16.19) and (16.17) we can obtain the following formula for the angular distribution of the radiated power:

$$\begin{aligned}\frac{dW}{d\Omega} &= \frac{e^2}{4\pi} \frac{[\mathbf{n} \times [(\mathbf{n}-\mathbf{v}) \times \mathbf{a}]]^2}{(1-\mathbf{n}\cdot\mathbf{v})^5} \\ &= \frac{e^2}{4\pi}\left\{\frac{\mathbf{a}^2}{(1-\mathbf{n}\cdot\mathbf{v})^3} + \frac{2(\mathbf{n}\cdot\mathbf{a})(\mathbf{v}\cdot\mathbf{a})}{(1-\mathbf{n}\cdot\mathbf{v})^4} - \frac{(1-v^2)(\mathbf{n}\cdot\mathbf{a})^2}{(1-\mathbf{n}\cdot\mathbf{v})^5}\right\},\end{aligned} \tag{16.20}$$

where the second equality was obtained through writing out the vector triple product in the numerator explicitly.

The radiation emitted by a particle also carries off momentum (see (15.13a)). We therefore wish to obtain a general expression for the 4-momentum $dP^\mu/d\tau d\Omega$ lost by the charge in the form of radiation emitted per unit proper time τ and per unit solid angle. It is expedient to carry out the calculations in four-dimensional form.

As shown in Section 15, the 4-momentum that passes through a closed surface σ during time dt is

$$dP^\mu = -dt \oint_\sigma T^{\mu k} n_k\, d\sigma, \tag{16.21}$$

where, as above, $d\sigma = r^2 d\Omega$, and $T^{\mu\nu}$ is the electromagnetic field energy-momentum tensor (15.5) (where the index "f" has been dropped).

Substituting into (15.5) the expression (16.9) for the field-strength tensor of the field created by the charge and performing simple

manipulations involving (16.5) and the fact that $w^\mu u_\mu = 0$, we arrive at the following formula for the energy-momentum tensor of the field generated by the charge:

$$T^{\mu\nu} = \frac{e^2}{(uX)^4}\left\{-\left[w^2 + \left(\frac{wX-1}{uX}\right)^2\right]X^\mu X^\nu + (X^\mu w^\nu + X^\nu w^\mu) - \frac{wX-1}{uX}(X^\mu u^\nu + X^\nu u^\mu) - \frac{1}{2}g^{\mu\nu}\right\}\Big|_{\tau_r}. \quad (16.22)$$

Let us move the surface σ in (16.21) into the wave zone. Then we can put (see (16.5)) $X^\mu = rn^\mu$, where

$$n^\mu = (1, \mathbf{n}), \quad (16.23)$$

and leave only the leading terms proportional to r^{-2} in tensor (16.22). The result is

$$T^{\mu\nu} \simeq -\frac{e^2}{4\pi r^2}\frac{n^\mu n^\nu}{(nu)^4}\left[w^2 + \left(\frac{nw}{nu}\right)^2\right]\Big|_{\tau_r}. \quad (16.22a)$$

Noting that the interval of proper time of the charge $d\tau_r \equiv d\tau = (1 - v^2)^{1/2}dt_r$ can be expressed, via (12.5), (16.18), and (16.23), in terms of the interval of the observer's time dt,

$$d\tau = dt_r/u^0 = dt/(nu), \quad (16.24)$$

we obtain, using (16.21) and (16.22a), the following formula for the radiated 4-momentum:

$$\frac{dP^\mu}{d\tau\, d\Omega} = -\frac{e^2}{4\pi}\frac{n^\mu}{(nu)^3}\left[w^2 + \left(\frac{nw}{nu}\right)^2\right]. \quad (16.25)$$

The temporal component of (16.25) can be reduced, if we go over from time τ to time t_r via (16.24) and employ (16.10), to the expression for the radiated power (16.20), while the spatial components are simply the radiation momentum:

$$\frac{d\mathbf{P}}{dt_r\, d\Omega} = \mathbf{n}\frac{d\mathcal{E}}{dt_r\, d\Omega}. \quad (16.25a)$$

Let us now find the total 4-momentum emitted per unit proper time in all directions:

$$\frac{dP^\mu}{d\tau} = \int \frac{dP^\mu}{d\tau\, d\Omega}\, d\Omega. \quad (16.26)$$

The angular integration in (16.26) can easily be carried out if we note that $dP^\mu/d\tau$ is a 4-vector (in contrast to (16.25), which is not)

and therefore must be proportional to the 4-vectors u^μ and w^μ that are at our disposal:

$$p^\mu \equiv \frac{dp^\mu}{d\tau} = -\frac{e^2}{4\pi}[Bu^\mu + Cw^\mu], \tag{16.27}$$

where B and C are still to be determined. Multiplying (16.27) scalarly into u_μ and employing (16.26) and (16.25) and the fact that $u^2 = 1$ and $(wu) = 0$, we get

$$B = Jw^2 + w_\mu w_\nu N^{\mu\nu}, \quad J = \int \frac{d\Omega}{(nu)^2}, \quad N^{\mu\nu} = \int \frac{n^\mu n^\nu}{(nu)^4}\, d\Omega. \tag{16.28}$$

Similarly, if we consider the scalar product (wp), we obtain

$$\begin{gathered} w^2 C = w^2 (w_\mu N^\mu) + w_\mu w_\nu w_\lambda N^{\mu\nu\lambda}, \\ N^\mu = \int \frac{n^\mu}{(nu)^3}\, d\Omega, \quad N^{\mu\nu\lambda} = \int \frac{n^\mu n^\nu n^\lambda}{(nu)^5}\, d\Omega. \end{gathered} \tag{16.29}$$

The tensor $N^{\mu\nu}$ in (16.28) has the form

$$N^{\mu\nu} = C_1 g^{\mu\nu} + C_2 u^\mu u^\nu.$$

We arrive at the following equations for C_1 and C_2:

$$g_{\mu\nu} N^{\mu\nu} = 4C_1 + C_2, \quad u_\mu u_\nu N^{\mu\nu} = C_1 + C_2 = J,$$

whence $C_1 = -J/3$ and

$$B = \frac{2}{3} Jw^2. \tag{16.30}$$

The tensors N^μ and $N^{\mu\nu\lambda}$ in (16.29) can also be expressed in terms of combinations of $g^{\mu\nu}$ and u^α, and, therefore, in view of the fact that $(wu) = 0$, it immediately follows that $C = 0$ in (16.27). Thus, we need only calculate the integral J:

$$J = u_0^{-2} \int_0^{2\pi} d\varphi \int_0^{\pi} \frac{d\theta \sin\theta}{(1 - v\cos\theta)^2} = \frac{2\pi}{u_0^2} \int_{-v}^{v} \frac{dx}{(1-x)^2} = 4\pi,$$

since $u_0 = (1 - v^2)^{-1/2}$. Substituting now (16.30) with $J = 4\pi$ into (16.27), we arrive at the final expression for the 4-momentum emitted per unit time:

$$\frac{dp^\mu}{d\tau} = -\frac{2}{3} e^2 w^2 u^\mu. \tag{16.31}$$

From this, for one, follows (if we allow for (16.24) and (16.10)) a formula for the total radiated power:

$$W \equiv \frac{d\varepsilon}{dt_r} = -\frac{2}{3} e^2 w^2 = \frac{2}{3} e^2 \left[\frac{\mathbf{a}^2}{(1-v^2)^2} + \frac{(\mathbf{a}\cdot\mathbf{v})^2}{(1-v^2)^3}\right], \tag{16.32}$$

which proves to be relativistically invariant (and, of course, positive in view of the fact that $w^2 = w^\mu w_\mu < 0$ (see (16.10)). Note that (16.32) can be obtained through integration with respect to angles of the three-dimensional expression (16.20), but this requires much more effort.

Thus, the total radiated power is the same in all inertial frames of reference. In particular, in the reference frame in which the particle is at rest ($\mathbf{v} = 0$) we obtain (in ordinary units)

$$W = \frac{2}{3c^3} e^2 \mathbf{a}^2, \tag{16.32a}$$

where $\mathbf{a}$ is the common three-dimensional acceleration. Therefore the radiation has a dipole nature: the dipole moment of the particle is $\mathbf{d} = e\mathbf{r}_e$, so that $W = (2/3c^3)\,\ddot{\mathbf{d}}^2$. Formula (16.32a) remains valid for particles moving with low velocities ($v/c \ll 1$) in the lowest order in parameter v/c. Note that dipole radiation is symmetric: the total momentum emitted per unit time is zero. The angular distribution of dipole radiation has the form (see (16.20) with $\mathbf{v} = 0$)

$$\frac{dW}{d\Omega} = \frac{e^2}{4\pi} [\mathbf{a} \times \mathbf{n}]^2 = \frac{e^2 \mathbf{a}^2}{4\pi} \sin^2 \theta, \tag{16.33}$$

where θ is the angle between $\mathbf{n}$ and $\mathbf{a}$.

The peculiarities of radiation emission by relativistic particles ($v \simeq c$) will be discussed in Section 17.

In conclusion we note that the applicability of the general formulas (16.25) and (16.31) is not limited by the fact that in their derivation we calculated the flux of 4-momentum in the wave zone. In the reference frame where $\mathbf{v} = 0$ only the wave term remains in (16.11) (for all distances R from the charge). Therefore, radiation can be observed not only in the wave zone (16.13) but in an arbitrary neighborhood of the charge as well.

16.3 The Radiation Spectrum

An important characteristic of radiation is the energy distribution over the frequencies of the monochromatic waves that comprise the field, or the spectral distribution of the radiation, which is commonly known as simply the spectrum.

Let us find the spectral expansion of the total emitted 4-momentum stemming from the law of conservation of the 4-momentum of the field-charges system in the form (15.9a). Integrating (15.9a) over the entire four-dimensional space and transforming the integral over the four-dimensional volume of the divergence of the energy-momentum tensor of the electromagnetic field via the divergence theo-

rem into an integral over an infinitely distant surface, we arrive at a formula for the total radiated 4-momentum:

$$P^{\mu} = -\int F^{\mu\nu}(x)\, j_{\nu}(x) d^4x, \tag{16.34}$$

which means that the radiated energy and momentum are (see (15.14) and (15.14a)) equal, respectively, to the work and impulse of the total Lorentz force acting on the charges in the system.

Expressing the field-strength tensor in terms of the 4-potential according to (13.1), $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$, and allowing for conservation of charge according to (10.18a), $\partial^{\nu}j_{\nu} = 0$, we find, after integration by parts, that the term $(\partial^{\nu}A^{\mu})\, j_{\nu}$ contributes nothing, so that (16.34) assumes the form

$$P^{\mu} = -\int (\partial^{\mu}A^{\nu})\, j_{\nu}\, d^4x. \tag{16.34a}$$

Now we expand the potential and current into four-dimensional Fourier integrals:

$$\begin{aligned} A^{\nu}(x) &= \int \frac{d^4k}{(2\pi)^4} e^{-ikx} A^{\nu}(k), \\ j_{\nu}(x) &= \int \frac{d^4k'}{(2\pi)^4} e^{-ik'x} j_{\nu}(k'). \end{aligned} \tag{16.35}$$

We substitute these expansions into (16.34a) and carry out integration with respect to coordinates and the four-dimensional wave vector k' via the well-known formula

$$\int d^4x e^{-i(k+k')x} = (2\pi)^4 \delta_4(k+k').$$

This yields

$$P^{\mu} = i\int \frac{d^4k}{(2\pi)^4} k^{\mu}\left(A^{\nu}_{\mathrm{ret}}(k)\, j_{\nu}(-k)\right), \tag{16.36}$$

and in view of the fact that the current $j^{\mu}(x)$ is real we get

$$j_{\nu}(-k) = j^{*}_{\nu}(k). \tag{16.37}$$

The wave equation (13.3) provides a link between the Fourier transforms of the potential $A^{\mu}(k)$ and the current $j^{\mu}(k)$:

$$A^{\mu}(k) = -\frac{4\pi}{k^2 \pm i\varepsilon k_0} j^{\mu}(k), \tag{16.38}$$

where the "plus" refers to the retarded potential A^{μ}_{ret} and the "minus" to the advanced potential A^{μ}_{adv} (see Section 13), with $\varepsilon \to +0$. Using (13.25), we can rewrite (16.38) in equivalent form:

$$A^{\mu}(k) = -4\pi j^{\mu}(k)\left[\mathcal{P}\frac{1}{k^2} \mp i\pi\, \mathrm{sgn}\, k_0 \delta(k^2)\right]. \tag{16.38a}$$

Substituting $A^{\mu}_{\text{ret}}(k)$ given by (16.38a) into (16.36), we get

$$P^{\mu} = -\frac{1}{(2\pi)^3}\int d^4k k^{\mu}\,\text{sgn}\,k^0\delta(x^2)\,(j^{\nu}(k)\,j_{\nu}(-k)). \qquad (16.39)$$

In the above derivation we allowed for the fact that the contribution of the principal value of the integral (see (16.38a)) is zero because the respective integrand is odd in k^{μ}, so that (16.39) is real, as it should be. Note also that in (16.39) we can replace $\text{sgn}\,k^0$ with $2\theta(k^0)$, where the theta function is defined by (13.16), since the integrand in (16.39) is even in k^0. As a result we arrive at the following spectral expansion of the radiated 4-momentum in final form:

$$P^{\mu} = -\frac{1}{2\pi^2}\int d^4k k^{\mu}\theta(k^0)\,\delta(k^2)\,[j^{\nu}(k)\,j^{*}_{\nu}(k)], \qquad (16.40)$$

here

$$k^{\mu} = (\omega, \mathbf{k}) \qquad (16.41)$$

is the four-dimensional wave vector, $\omega \equiv k^0 = |\mathbf{k}| \geqslant 0$, which is ensured by the presence of the theta and delta functions in (16.40).

As the reader can see from the above derivation, the contribution to the radiation is provided only by the half-difference of the retarded and advanced potentials (see (16.38a)):

$$A^{\mu}_{\text{rad}}(k) \equiv \frac{i}{2}[A^{\mu}_{\text{ret}}(k) - A^{\mu}_{\text{adv}}(k)] = i(2\pi)^2\,\text{sgn}\,k^0\delta(k^2)\,j^{\mu}(k), \qquad (16.42)$$

or, in the coordinate representation (see (16.35) and the transition from (16.39) to (16.40)),

$$A^{\mu}_{\text{rad}}(x) \equiv \frac{1}{2}[A^{\mu}_{\text{ret}}(x) - A^{\mu}_{\text{adv}}(x)]$$
$$= \frac{i}{(2\pi)^3}\int d^4k\theta(k^0)\,\delta(k^2)\,j^{\mu}(k)\,e^{-ikx}. \qquad (16.42a)$$

In this manner, the radiation produced by a moving charge (the electromagnetic field) can be represented in the form of an expansion in plane monochromatic waves (see also (13.19) and (13.26)) with frequencies ω and wave vectors $\mathbf{k} = \omega\mathbf{n}$, where $\mathbf{n}$ is the unit vector pointing in the direction of propagation of the wave. The frequency and wave vector of each wave form a 4-vector (16.41).

The Fourier transform of $j^{\nu}(k)$ in (16.40) is defined, in accord with (16.35), by the formula

$$j^{\nu}(k) = \int d^4x e^{ikx} j^{\nu}(x). \qquad (16.43)$$

For a single charged particle, the current $j^\nu(x)$ has the form (12.4), and the Fourier transform of this current can easily be found via (16.43):

$$j^\nu(k) = e\int_{-\infty}^{\infty} d\tau u^\nu(\tau)\, e^{ikz(\tau)}. \tag{16.44}$$

Note that here $z^\mu(\tau)$ is the four-dimensional radius vector of the particle, $u^\mu = dz^\mu/d\tau$ is the four-dimensional velocity, and integration is carried out with respect to the proper time τ.

Let us use (16.40) to find the three-dimensional expression for the total energy $d\varepsilon(\mathbf{n}, \omega)$ radiated (over the entire time of the radiation process being investigated) into the solid angle element $d\Omega$ in the direction $\mathbf{n}$ in the form of waves with frequencies lying in the ω to $\omega + d\omega$ frequency range. Taking the temporal component of (16.40), integrating it with respect to k^0 via

$$\theta(k^0)\,\delta(k^2) = \frac{1}{2\omega}\,\delta(k^0 - \omega)\cdot \tag{16.45}$$

(see (13.14)), and writing the volume element in the $\mathbf{k}$-space as $d^3k = \omega^2\, d\omega\, d\Omega$, we arrive at the following formula for the angular spectral distribution of the radiation energy:

$$\frac{d\varepsilon(\mathbf{n}, \omega)}{d\Omega\, d\omega} = \frac{\omega^2}{(2\pi)^3}\,|j^\mu(k)|^2, \tag{16.46}$$

where for the sake of brevity we have introduced the notation $|j^\mu|^2 = -(j^\mu(k)\, j^*_\mu(k))$.

The radiation spectrum of a single particle follows from (16.46) and (16.44):

$$\frac{d\varepsilon}{d\Omega\, d\omega} = \frac{e^2\omega^2}{(2\pi)^3}\left|\int_{-\infty}^{\infty} d\tau u^\mu(\tau)\, e^{ikz(\tau)}\right|^2, \tag{16.47}$$

or after we go over to the laboratory time t,

$$\begin{aligned}\frac{d\varepsilon}{d\Omega\, d\omega} &= \frac{e^2\omega^2}{(2\pi)^3}\left|\int_{-\infty}^{\infty} dt\, v^\mu(t)\exp\left[i\omega t - i\,\mathbf{k}\cdot\mathbf{r}_e(t)\right]\right|^2 \\ &= \frac{e^2\omega^2}{(2\pi)^3}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\left[\mathbf{v}(t_1)\cdot\mathbf{v}(t_2) - 1\right]\exp\left\{i\omega(t_1 - t_2)\right. \\ &\quad \left. - i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)\right\}. \end{aligned} \tag{16.47a}$$

Here we have introduced the notations $v^\mu(t) = u^\mu/u^0 = (1, \mathbf{v}(t))$, $\mathbf{r}_1 = \mathbf{r}_e(t_1)$, and $\mathbf{r}_2 = \mathbf{r}_e(t_2)$.

Up till now we considered only the spectral expansion of the total radiation energy. This quantity is important in problems on particle collisions and when the particles fly through regions occupied by an external field. The corresponding radiation is commonly known as bremsstrahlung (braking radiation).

In the case of periodic motion of particles (say, electrons in a synchrotron or storage ring), the radiation energy grows without bound in time, and there appears the question of the spectral distribution of the radiated power (or intensity) W averaged over the period T of the motion:

$$\langle W\rangle = \frac{1}{T}\int_0^T W(t)\,dt.$$

Let us find the spectrum of the radiated power from the spectrum of the total radiation energy (16.47a). But first let us transform the Fourier transform of an arbitrary periodic function $f(t) = f(t+T)$ (T is the period):

$$f(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{i\omega t}\,dt = \sum_{l=-\infty}^{\infty} e^{il\omega T}\int_0^T f(t)\,e^{i\omega t}\,dt,$$

or, using the well-known formula

$$\sum_{l=-\infty}^{\infty} e^{il\psi} = 2\pi\sum_{l=-\infty}^{\infty}\delta(\psi - 2\pi l),$$

we get

$$f(\omega) = 2\pi\sum_{l=-\infty}^{\infty} f_l\,\delta(\omega - l\omega_0), \tag{16.48}$$

where

$$f_l = \frac{1}{T}\int_0^T f(t)\,e^{il\omega_0 t}\,dt,$$

and $\omega_0 = 2\pi/T$ is the frequency of the periodic motion.

Next, from (16.48) we get

$$|f(\omega)|^2 = (2\pi)^2\sum_{l,\,l'} f_l f_{l'}^*\,\delta(\omega - l\omega_0)\,\delta(\omega - l'\omega_0)$$

$$= (2\pi)^2\,\delta(0)\sum_{l=-\infty}^{\infty}|f_l|^2\,\delta(\omega - l\omega_0), \tag{16.49}$$

where the (infinite) value of the delta function, $\delta(0)$, is understood in the limiting sense:

$$2\pi\delta(0) = \lim_{T_0\to\infty} \lim_{\omega\to 0} \int_{-T_0/2}^{T_0/2} e^{i\omega t}\, dt = \lim_{T_0\to\infty} T_0,$$

with T_0 understood to be the total time during which radiation occurs. This leads us to the following relationship that links the spectrum of the radiated power, $dW(\omega)$, and the energy spectrum $d\varepsilon(\omega)$:

$$dW(\omega) = \lim_{T_0\to\infty} d\varepsilon(\omega)/T_0.$$

Substituting the expression (16.47) transformed via (16.49), we find the angular spectral distribution of the radiated power:

$$\frac{dW}{d\Omega\, d\omega} = \frac{e^2\omega^2}{2\pi} \sum_{l=1}^{\infty} \delta(\omega - l\omega_0) \left| \frac{1}{T} \int_0^T dt\, v^\mu(t) \exp\left[il\omega_0 (t - \mathbf{n}\cdot\mathbf{r}_e(t))\right] \right|^2. \tag{16.50}$$

Here we have taken into account the fact that ω is positive, so that the terms with $l \leqslant 0$ contribute nothing to the final result.

According to (16.50), in the case of periodic motion the particle emits a discrete set of waves with frequencies $\omega_l = l\omega_0$, which are harmonics of the fundamental frequency of the motion, $\omega_0 = 2\pi/T$.

The expression for the radiated power of the lth harmonic follows from (16.40) if we integrate with respect to frequencies ω:

$$\begin{aligned}\frac{dW_l}{d\Omega} &= \frac{e^2 l^2}{(2\pi)^3}\omega_0^4 \left| \int_0^T dt\, v^\mu(t) \exp\left[il\omega_0 (t - \mathbf{n}\cdot\mathbf{r}_e(t))\right] \right|^2 \\ &= \frac{e^2 l^2}{(2\pi)^3}\omega_0^4 \int_0^T dt_1 \int_0^T dt_2 \left[\mathbf{v}(t_1)\cdot\mathbf{v}(t_2) - 1\right] \\ &\times \exp\left\{il\omega_0 \left[(t_1 - t_2) - \mathbf{n}\cdot(\mathbf{r}_e(t_1) - \mathbf{r}_e(t_2))\right]\right\}. \end{aligned} \tag{16.51}$$

Note that the spectral formulas were obtained without resorting to the concept of a wave zone.

16.4 Polarization of the Emitted Radiation

The radiation field is determined by the field vectors **E** and **H** lying in the plane perpendicular to the direction of propagation of the radiation, **n**, with the three vectors forming a right-handed triple (see Section 16.1): $\mathbf{E} \perp \mathbf{H}$ and $\mathbf{n} = (\mathbf{E} \times \mathbf{H})/EH$. The polarization of the radiation usually characterizes the direction of vector **E**.

The polarization properties of radiation can be described in the following manner. We decompose the Fourier transform of the electric-field vector, $\mathbf{E}(\omega)$, into two mutually orthogonal components lying in the plane perpendicular to $\mathbf{n}$:

$$\mathbf{E}(\omega) = \sum_{\lambda} E_{\lambda}(\omega)\, \mathbf{e}_{(\lambda)}, \tag{16.52}$$

where $\mathbf{e}_{(\lambda)}$ ($\lambda = 1, 2$) are generally complex-valued vectors satisfying the conditions

$$\mathbf{n} \cdot \mathbf{e}_{(\lambda)} = 0, \quad \mathbf{e}^{*}_{(\lambda')} \cdot \mathbf{e}_{(\lambda)} = \delta_{\lambda'\lambda}. \tag{16.53}$$

We remind the reader that the real-valued vectors $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$ fix the linearly polarized components of the radiation, while the complex-valued combinations

$$\mathbf{e}_{(\pm)} = \frac{1}{\sqrt{2}} (\mathbf{e}_{(1)} \pm i\mathbf{e}_{(2)})$$

correspond to components with circular polarization (with opposite senses of rotation of vector $\mathbf{E}$). The polarizaton state can be defined by studying the component ratio $E_{\lambda}/E_{\lambda'}$.

If polarization is described in four-dimensional language, the expansion (16.52) corresponds to the expansion of the Fourier transform of the 4-potential (see (16.35)):

$$A^{\mu}(k) = \sum_{\lambda} e^{\mu}_{(\lambda)} A_{(\lambda)}(k), \tag{16.54}$$

with

$$ke_{(\lambda)} = 0, \; e^{*}_{(\lambda)}\, e_{(\lambda)} = -1. \tag{16.55}$$

The mutual orthogonality of k and $e_{(\lambda)}$ follows from the Lorentz condition (see (13.19) and (13.20)). Let us see how polarization 4-vectors $e^{\mu}_{(\lambda)}$ are transformed under gauge transformations (10.9). The radiation field is represented in the form of an expansion in plane monochromatic waves (see (16.42a)); the gauge function, which in Lorentz gauge obeys the homogeneous wave equation (see (10.23)), can also be represented by a similar expansion:

$$\chi(x) = \int \frac{d^4k}{(2\pi)^4}\, i\tilde{\chi}(k)\, \theta(k^0)\, \delta(k^2)\, e^{-ikx}. \tag{16.56}$$

From (10.9) and (16.56) follows the gauge transformation of the polarization 4-vector

$$e^{\mu}_{(\lambda)} \to e'^{\mu}_{(\lambda)} = e^{\mu}_{(\lambda)} + \tilde{\chi}(k)\, k^{\mu}, \tag{16.57}$$

with $\tilde{\chi}(k)$ an arbitrary function of k. Since $k^2 = 0$, the above transformation preserves orthogonality and normalization:

$$ke'_{(\lambda)} = 0, \; e'^{*}_{(\lambda)}\, e'_{(\lambda)} = -1. \tag{16.55a}$$

In any fixed reference frame, the function $\tilde{\chi}(k)$ in (16.57) can be selected in such a manner that it nullifies the temporal component $e^0_{(\lambda)}$, that is,

$$e^{\mu}_{(\lambda)} = (0, \mathbf{e}_{(\lambda)}) \text{ and } e^*_{(\lambda)} e_{(\lambda)} = -\mathbf{e}^*_{(\lambda)} \cdot \mathbf{e}_{(\lambda)} = -1.$$

This agrees with (16.53) and explains the choice of the four-dimensional normalization condition in the form (16.55), which ensures that the polarization 4-vector $e_{(\lambda)}$ is spacelike (see Section 2).

The separate components of the field with different polarization λ (see (16.52) and (16.54)) provide, obviously, independent contributions to the radiation. These contributions can easily be found if we note that $A_{(\lambda)}(k) = [e^{*\mu}_{(\lambda)} A_{\mu}(k)]$ in view of (16.54) and $e^*_{(\lambda')} e_{(\lambda)} = 0$ at $\lambda' \neq \lambda$ and allow for the relationship that links $A^{\mu}(k)$ with the Fourier transform of the current, $j^{\mu}(k)$, in the form (16.42). Note, for instance, that in terms of Fourier transforms the conservation of charge expressed by Eq. (10.18a), $\partial_{\mu} j^{\mu}(x) = 0$, implies the orthogonality of the 4-vectors k and $j(k)$:

$$k^{\mu} j_{\mu}(k) = 0. \tag{16.58}$$

The radiation-field potential (16.42a), of course, agrees with (16.58) in Lorentz gauge.

The above remarks imply that a formula for the 4-momentum $P^{\mu}_{(\lambda)}$ of a radiation with a certain polarization λ can be obtained from (16.40) by replacing $j^{\nu}(k)$ with $e^*_{(\lambda)} j(k)$:

$$P^{\mu}_{(\lambda)} = \frac{1}{2\pi^2} \int d^4k\, k^{\mu} \theta(k^0)\, \delta(k^2)\, |\, e^*_{(\lambda)} j(k)\, |^2. \tag{16.59}$$

The angular spectral distribution of the energy of the polarized radiation produced by a single particle follows from (16.47) and (16.47a) after similar replacements:

$$\frac{d\mathcal{E}_{(\lambda)}}{d\Omega\, d\omega} = \frac{e^2\omega^2}{(2\pi)^2} \left| \int_{-\infty}^{\infty} d\tau\, (e^*_{(\lambda)} u(\tau))\, e^{ikz(\tau)} \right|^2, \tag{16.60}$$

or, in three-dimensional form,

$$\frac{d\mathcal{E}_{(\lambda)}}{d\Omega\, d\omega} = \frac{e^2\omega^2}{(2\pi)^2} \left| \int_{-\infty}^{\infty} dt\, (\mathbf{e}^*_{(\lambda)} \cdot \mathbf{v}(t)) \exp[i\omega t - i\mathbf{k} \cdot \mathbf{r}_e(t)] \right|^2$$

$$= \frac{e^2\omega^2}{(2\pi)^2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2\, (\mathbf{e}^*_{(\lambda)} \cdot \mathbf{v}(t_1))\, (\mathbf{e}_{(\lambda)} \cdot \mathbf{v}(t_2)) \exp\{i\omega[(t_1 - t_2)$$

$$- \mathbf{n} \cdot (\mathbf{r}_1 - \mathbf{r}_2)]\}. \tag{16.60a}$$

In the case of a periodic motion of the particle, Eq. (16.51) yields the following expression for the radiated power of the lth harmonic with polarization λ:

$$\frac{dW_l^{(\lambda)}}{d\Omega} = \frac{e^2 l^2}{(2\pi)^3}\omega_0^4 \left| \int_0^T dt\,(\mathbf{e}^*_{(\lambda)}\cdot\mathbf{v}(t)) \exp\left[il\omega_0 (t - \mathbf{n}\cdot\mathbf{r}_e(t))\right]\right|^2. \quad (16.61)$$

Finally, let us consider the expression for the instantaneous radiated power of polarized radiation. To this end, we must replace $\mathbf{E}^2$ with E_λ^2 in (15.13), where $E_\lambda = \mathbf{e}_{(\lambda)}\cdot\mathbf{E}$, or in four-dimensional notation,

$$E_\lambda = e^\mu_{(\lambda)} F_{0\mu},$$

with the polarization vector $e_{(\lambda)}$ being, obviously, real-valued in view of the fact that $\mathbf{E}(t, \mathbf{r})$ is real-valued.

Let us go over to the wave zone and retain only the leading terms proportional to r^{-1} in the field-strength tensor (16.9). Then the component with polarization λ in the electric field assumes the form

$$E_\lambda = \frac{1}{r}\frac{e}{(nu)^3}\left[e_{(\lambda)}\left(w - \frac{(nw)}{(nu)}u\right)\right],$$

where we have employed the same notation as in (16.9) and (16.23). All further calculations are similar to those that led to formula (16.25). As a result we arrive at the following expression for the angular distribution of the 4-momentum of radiation of polarization λ and emitted per unit proper time of the particle:

$$\frac{dP^\mu_{(\lambda)}}{d\tau\, d\Omega} = \frac{e^2}{4\pi}\frac{n^\mu}{(nu)^3}\left[e_{(\lambda)}\left(w - \frac{nw}{nu}u\right)\right]^2. \quad (16.62)$$

If we sum over the two independent polarization states, we will arrive at (16.25), as expected.

Combining (16.62) with (12.6), (16.10), and (16.24), we find the following expression for the angular distribution of the radiated power for polarized radiation (in three-dimensional notation):

$$\frac{dW^{(\lambda)}}{d\Omega} = \frac{e^2}{4\pi}(1 - \mathbf{n}\cdot\mathbf{v})^{-5}[(\mathbf{e}_{(\lambda)}\cdot\mathbf{a})(1 - \mathbf{n}\cdot\mathbf{v}) + (\mathbf{e}_{(\lambda)}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{a})]^2. \quad (16.63)$$

This formula can readily be obtained from (16.20) by replacing $\mathbf{n}\times((\mathbf{n} - \mathbf{v})\times\mathbf{a})$ with $\mathbf{e}_{(\lambda)}\cdot[\mathbf{n}\times((\mathbf{n} - \mathbf{v})\times\mathbf{a})]$, writing out explicitly the scalar triple product, and allowing for the fact that

$\mathbf{e}_{(\lambda)} \cdot \mathbf{n} = 0$. Summing in (16.63) over the polarizations λ via the relationship*

$$\sum_{\lambda} e^i_{(\lambda)} e^k_{(\lambda)} = \delta^{ik} - n^i n^k, \tag{16.64}$$

we arrive at the formula (16.20) for the radiated power of nonpolarized radiation.

We see that the formulas obtained in this section make it possible to calculate the main characteristics of the radiation generated by an arbitrarily moving charged particle: the radiated power, the angular and spectral distributions of the radiation, and the polarization of the radiation.

17. Synchrotron Radiation

Let us apply the results of the previous section to the study of synchrotron radiation, which is generated by ultrarelativistic electrons moving in a magnetic field (e.g. moving in synchrotrons or storage rings.)**

17.1 Angular Spectral Distribution of Radiated Power of Synchrotron Radiation Emitted by Electrons Moving with Arbitrary Velocity

In a uniform magnetic field $\mathbf{H} = H\mathbf{e}_z$, the electron generally moves along a helical trajectory with a constant energy $\varepsilon = m(1 - v^2)^{-1/2}$ and a projection $v_{||}$ of the velocity vector on $\mathbf{H}$. This case ($v_{||} \neq 0$), which is important to astrophysics, is considered in [2]. We will restrict our discussion to electron motion along a circular orbit (with $v_{||} = 0$), motion that is realized in a cyclic accelerator. We select the coordinate origin at the center of the orbit of radius R and write the law of electron motion in the form

$$\mathbf{r}_e(t) = R\mathbf{e}_x \cos \omega_0 t + R\mathbf{e}_y \sin \omega_0 t, \tag{17.1}$$

where the cyclic frequency ω_0 is given by the formula

$$\omega_0 = eH/\varepsilon = v/R. \tag{17.2}$$

* The right-hand side of (16.64) is determined uniquely by the following conditions: it must be a symmetric (three-dimensional) tensor composed of the unit tensor δ^{ik} and the components of vector $\mathbf{n}$, its trace is equal to 2 (in view of (16.53)), and the contraction of this tensor with n^i or n^k yields zero.

** A detailed discussion of the theory of this important physical phenomenon can be found in [2]. A review of the general properties of synchrotron radiation and the results of their experimental study can be found in Section 36.

In view of the periodicity of (17.1), the radiation spectrum is discrete with allowed frequencies

$$\omega_\nu = \nu\omega_0 \quad (\nu = 1, 2, 3, \ldots), \tag{17.3}$$

which are harmonics of ω_0 (17.2). The angular distribution of the radiated power of the νth harmonic with a given polarization λ is given by the general formula (16.61) with $\mathbf{r}_e(t)$ taken in the form (17.1).

To characterize polarized radiation, we introduce the unit vectors for the σ- and the π-component of linear polarization, namely

$$\mathbf{e}_\sigma = \frac{\mathbf{n}\times\mathbf{e}}{|\mathbf{n}\times\mathbf{e}|}, \quad \mathbf{e}_\pi = \mathbf{n}\times\mathbf{e}_\sigma, \tag{17.4}$$

where

$$\mathbf{n} = \mathbf{e}_x \sin\theta\cos\varphi + \mathbf{e}_y \sin\theta\sin\varphi + \mathbf{e}_z\cos\theta \tag{17.5}$$

is a unit vector pointing in the direction in which the radiation is propagating (specified by angles θ and φ in a spherical system of coordinates), and $\mathbf{e} = \mathbf{H}/H = \mathbf{e}_z$ is a unit vector pointing in the direction of the magnetic field. Decomposing the unit vectors (17.4) in Cartesian unit vectors, we find that

$$\begin{aligned} \mathbf{e}_\sigma = \mathbf{e}_\varphi &= -\mathbf{e}_x \sin\varphi + \mathbf{e}_y\cos\varphi, \\ \mathbf{e}_\pi = -\mathbf{e}_\theta &= -\mathbf{e}_x\cos\theta\cos\varphi - \mathbf{e}_y\cos\theta\sin\varphi + \mathbf{e}_z\sin\theta. \end{aligned} \tag{17.4a}$$

As shown by (16.61), the calculation of the radiation properties is reduced to evaluating integrals (proportional, obviously, to the Fourier transforms of the radiation electric field, $E_\lambda(\nu)$):

$$v_\lambda(\nu) = \int_0^T dt\, v_\lambda(t)\, e^{i\nu\omega_0 t'}, \tag{17.6}$$

where $T = 2\pi/\omega_0$ is the period of the electron's orbital motion,

$$t' = t - \mathbf{n}\cdot\mathbf{r}_e(t), \tag{17.7}$$

and

$$v_\lambda(t) = (\mathbf{e}_\lambda\cdot\mathbf{v}(t)) \quad (\lambda = \sigma, \pi) \tag{17.8}$$

are the projections of the velocity $\mathbf{v}(t) = d\mathbf{r}_e/dt$ on the unit vectors (17.4a). If we employ (17.1), (17.4a), and (17.5), we can write (17.7) and (17.8) in explicit form:

$$\begin{aligned} t' &= t - R\sin\theta\sin\alpha, \quad \alpha = \omega_0 t + \frac{\pi}{2} - \varphi, \\ v_\sigma(t) &= v\sin\alpha, \quad v_\pi(t) = -v\cos\theta\cos\alpha. \end{aligned} \tag{17.9}$$

We will now use the well-known expansions:

$$\begin{pmatrix} 1 \\ \sin\alpha \\ \cos\alpha \end{pmatrix} e^{-ix\sin\alpha} = \sum_{l=-\infty}^{\infty} \begin{pmatrix} J_l(x) \\ iJ_l'(x) \\ \frac{l}{x} J_l(x) \end{pmatrix} e^{-il\alpha}, \tag{17.10}$$

where $J_l(x)$ is the Bessel function of the first kind, and $J_l' = dJ_l(x)/dx$ is its derivative. Then integration with respect to time t in (17.6) is reduced to

$$\int_0^T dt \exp[iv\omega_0 t - il\alpha] = \frac{2\pi}{\omega_0} e^{-iv\left(\frac{\pi}{2}-\varphi\right)} \delta_{lv}, \tag{17.11}$$

where δ_{lv} is the Kronecker delta, and we have used the formula (17.9) for α. As a result, (17.6) takes the form

$$\begin{pmatrix} v_\sigma(\nu) \\ v_\pi(\nu) \end{pmatrix} = \frac{2\pi}{\omega_0} e^{-iv\left(\frac{\pi}{2}-\varphi\right)} \begin{pmatrix} ivJ_\nu'(\xi) \\ -\cot\theta J_\nu(\xi) \end{pmatrix}, \tag{17.12}$$

where the argument of the Bessel function is

$$\xi = \nu v \sin\theta. \tag{17.13}$$

Substituting (17.12) into (16.61), we arrive at a formula for the angular spectral distribution of the radiated power of the σ- and the π-component:

$$\begin{pmatrix} dW_\sigma(\nu) \\ dW_\pi(\nu) \end{pmatrix} = \frac{e^2\nu^2\omega_0^2}{2\pi} d\Omega \begin{pmatrix} v^2 J_\nu'^2(\xi) \\ \cot^2\theta J_\nu^2(\xi) \end{pmatrix}. \tag{17.14}$$

Here the solid angle element $d\Omega = \sin\theta\, d\theta\, d\varphi$ can be replaced, after we integrate with respect to φ, with $2\pi \sin\theta\, d\theta$, since $dW_\lambda/d\Omega$ is independent of angle φ, whereby the total radiated power $W = W_\sigma + W_\pi$ is given by the formula

$$W = \sum_{\nu=1}^{\infty} W(\nu) = e^2\omega_0^2 \sum_{\nu=1}^{\infty} \nu^2 \int_0^\pi d\theta \sin\theta\, [v^2 J_\nu'^2(\xi) + \cot^2\theta J_\nu^2(\xi)], \quad \xi = \nu v \sin\theta, \tag{17.15}$$

where the term $W(\nu)$ corresponds to the radiated power of the νth harmonic. Formula (17.15) is known as the Schott formula.

Formulas (17.14) and (17.15) are valid for all values of velocity v. In the nonrelativistic limit $v \ll 1$, the argument ξ of the $J_\nu(\xi)$ is small. Therefore, we can restrict our discussion to the first terms in the series expansions of these functions: $J_\nu(\xi) \simeq (\xi/2)^\nu/\nu!$. From this it follows that the main contribution to the total radiated power

(17.15) is provided by the first harmonic ($\nu = 1$, dipole radiation), with the radiated power of this harmonic given by the formula

$$W(1) = \frac{2}{3}\frac{e^2 v^4}{R^2} \quad (v \ll 1). \tag{17.16}$$

The radiation of the higher harmonics ($\nu > 1$) is suppressed in a power fashion: $W(\nu)/W(1) \propto v^{2(\nu-1)} \ll 1$.

Most interesting from the viewpoint of applications is the ultrarelativistic case, $v \simeq 1$, that is, the electron energy ε is much higher than m. The maximum in the radiation spectrum lies in the region of high harmonics, $\nu \sim (\varepsilon/m)^3$ (see below). For this reason, the Schott formula (17.15), in which now both the orders ν and the arguments ξ of the Bessel functions $J_\nu(\xi)$ are large, proves to be unsuitable for practical applications. However, it is possible to obtain convenient approximate formulas describing the radiation emitted by ultrarelativistic particles.

17.2 Peculiarities of Radiation Emitted by Relativistic Particles

Let us consider in greater detail the radiation emitted by highly energetic particles, that is, particles for which

$$\gamma = \frac{\varepsilon}{m} = (1 - v^2)^{-1/2} \gg 1. \tag{17.17}$$

The angular distribution of the radiated power of a particle moving in an arbitrary manner is given by formula (16.20). For ultrarelativistic particles ($\gamma^{-2} \ll 1$) this distribution has a special feature: because (16.20) contains high negative powers of $1 - \mathbf{n}\cdot\mathbf{v}$, the radiation is limited mainly to a narrow cone with the axis directed along $\mathbf{v}$. The angle at the vertex of the cone is determined, in order of magnitude, by the approximate relationship

$$1 - \mathbf{n}\cdot\mathbf{v} = 1 - v\cos\vartheta \simeq 1 - v + \frac{\vartheta^2}{2} \simeq \frac{1}{2}(\gamma^{-2} + \vartheta^2), \tag{17.18}$$

where ϑ is the angle between $\mathbf{n}$ and $\mathbf{v}$. Hence, the angle at the vertex, ϑ, obeys the following condition:

$$\vartheta \lesssim \gamma^{-1} = m/\varepsilon, \tag{17.19}$$

since outside this range the radiated power falls off rapidly.

The shape of the radiation spectrum is determined by the ratio of the characteristic angle γ^{-1} to the total angle $\Delta\vartheta$ by which the vector $\mathbf{v}$ is deflected during the entire process of motion of the particle (say, the time of flight of the particle through the region occupied by an external field) or during one period of the motion (in the case of periodic motion, say, in a cyclic accelerator).

There are two characteristic limiting cases:

(1) $\Delta\vartheta \gg \gamma^{-1}$. Here the radiation in a given direction **n** (precisely, the radiation emitted into a cone with angle γ^{-1} at the vertex and with axis directed along **n**) is determined by the portion of the trajectory over which the velocity vector **v** rotates through an angle of the order of γ^{-1}. The length of this portion (by order of magnitude) is

$$l_r = R/\gamma \tag{17.20}$$

(where R is the radius of curvature of the trajectory) and is known as the radiation formation length. Synchrotron radiation is a typical example. If the external field in which the particle moves remains practically the same over l_r, all the characteristics of the radiation have a universal nature, that is, are valid for any external field (provided its inhomogeneity scale is greater than l_r) and are determined by the instantaneous values of particle velocity **v** and acceleration **a** [11]. Therefore, the formulas we will obtain in Section 17.3 remain valid in the more general case ($\Delta\vartheta \gg \gamma^{-1}$).

(2) $\Delta\vartheta \ll \gamma^{-1}$. Here the radiation emitted in a given direction is collected from the entire trajectory and therefore depends on the nature of the external field. An example is the undulator radiation, which we discuss in brief in Section 36.

Let us now discuss qualitatively the radiation spectrum in case (1), that is, in particular, the synchrotron-radiation spectrum. The above analysis implies that a short pulse of radiation is observed in a given direction **n** (or a series of such pulses if the particle's motion is periodic). Let us evaluate the duration (or length) of this pulse. According to (17.20), the particle emits along a fixed direction, precisely, into a cone with an angle of order of γ^{-1} at the vertex of the cone, in the course of a time interval of the order of

$$\Delta t_r = l_r/v \simeq R/\gamma, \tag{17.21}$$

since $v \simeq 1$. The observer registers the radiation in the course of the interval (see (16.18) and (17.18))

$$\Delta t = (1 - \mathbf{n}\cdot\mathbf{v})\,\Delta t_r \sim \gamma^{-2}\,\Delta t_r,$$

or, if we allow for (17.21) and (17.2), where we can put $v \simeq 1$,

$$\Delta t \sim 1/\gamma^3\omega_0, \tag{17.22}$$

which provides an estimate for the pulse length. In the theory of Fourier transformations it is shown that a pulse of duration Δt is expandable into a spectrum extending to frequencies of the order of

$$\omega_c \sim 1/\Delta t \sim \gamma^3\omega_0. \tag{17.23}$$

As a rule, at low frequencies the radiated power grows with ω, and therefore (17.23) provides an estimate for the position of the maximum in the radiation spectrum.

Thus, ultrarelativistic particles emit radiation with frequencies of the order of ω_c (17.23), which are higher harmonics $\nu \sim \gamma^3$ of the characteristic frequency ω_0 of the motion.

The short duration of the radiation pulse, $\Delta t \ll 1/\omega_0$, emitted by an ultrarelativistic particle makes it possible to introduce the concept of the instantaneous spectrum of the radiated power, a quantity determined by the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ at a given moment t as follows. The angular spectral distribution of the energy of radiation of polarization λ is given by formula (16.60a) in the form of a double integral with respect to t_1 and t_2. It is convenient to introduce new variables of integration t and t_0 via the following formulas:

$$t_1 = t - t_0/2, \quad t_2 = t + t_0/2. \tag{17.24}$$

Then (16.60a) is reduced to an integral of the type

$$\frac{d\mathcal{E}^{(\lambda)}}{d\Omega\, d\omega} = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_0 f(t - t_0/2,\ t + t_0/2),$$

where the integrand in the integral with respect to t (that is, the integral with respect to t_0) can be considered the angular spectral distribution of the radiated power at time t:

$$\frac{dW_\lambda}{d\Omega\, d\omega} = \frac{d\mathcal{E}^{(\lambda)}/dt}{d\Omega\, d\omega}$$

$$= \frac{e^2\omega^2}{(2\pi)^2} \int_{-\infty}^{\infty} dt_0\, (\mathbf{e}_\lambda^* \cdot \mathbf{v}_1)\, (\mathbf{e}_\lambda \cdot \mathbf{v}_2) \exp\{i\omega[(t_1 - t_2) - \mathbf{n}\cdot(\mathbf{r}_1 - \mathbf{r}_2)]\}, \tag{17.25}$$

$$\mathbf{v}_{1,2} = \mathbf{v}(t \mp t_0/2), \quad \mathbf{r}_{1,2} = \mathbf{r}_e(t \mp t_0/2).$$

In view of the smallness of the radiation pulse formation time (see (17.21)), the main contribution to the integral with respect to t_0 is provided by the region of small t_0, precisely, such values of t_0 for which $|\Delta\mathbf{v}| = |\mathbf{v}_2 - \mathbf{v}_1| \sim \omega_0 t_0 \sim \gamma^{-1} \ll 1$. Therefore, we must expand the main quantities that determine the spectrum (17.25) in power series in t_0 ($\gamma^{-1} \ll 1$ is the dimensionless expansion parameter). But first let us introduce new angles that specify the direction $\mathbf{n}$ of the propagation of radiation (instead of θ and φ in (17.5)), namely, ψ, the angle between $\mathbf{n}$ and the plane $(\mathbf{v}, \mathbf{a})$ (the orbital plane in the case of synchrotron radiation, when $\psi = \pi/2 - \theta$), and δ, the angle between $\mathbf{v}$ and the projection of $\mathbf{n}$ on this plane. Such a choice is convenient because the principal contribution to the radiation is provided by the region of small angles $\psi \lesssim \gamma^{-1}$, $\delta \lesssim \gamma^{-1}$,

as demonstrated by (17.19). The definition of ψ and δ yields the following approximate formula (to within terms of the order of γ^{-2}):

$$\mathbf{n}\cdot\mathbf{v} = v\cos\psi\cos\delta \simeq v\left(1-\frac{\psi^2+\delta^2}{2}\right),$$
$$1-\mathbf{n}\cdot\mathbf{v} \simeq \frac{1}{2}(\gamma^{-2}+\psi^2+\delta^2), \quad \mathbf{n}\cdot\mathbf{a} = a\cos\psi\sin\delta \simeq a\delta, \tag{17.26}$$
$$d\Omega = d\psi\, d\delta.$$

Here we assume that the acceleration $\mathbf{a} = d\mathbf{v}/dt$ is perpendicular to the velocity, and in the case of synchrotron radiation

$$\mathbf{a} = \frac{e}{\varepsilon}\mathbf{v}\times\mathbf{H}, \quad \frac{d\mathbf{a}}{dt} = -\omega_0^2\mathbf{v}. \tag{17.27}$$

If we employ (17.26), (17.27), and the definition of the unit polarization vectors (17.4), $\mathbf{e}_\lambda$, we can easily find the expansions for the quantities $\mathbf{e}_\lambda\cdot\mathbf{v}_{1,2}$ and the expression in the square brackets in (17.25):

$$\mathbf{v}_{1,2}\cdot\mathbf{e}_\sigma \simeq -\delta \pm \frac{1}{2}\omega_0 t_0, \quad \mathbf{v}_{1,2}\cdot\mathbf{e}_\pi \simeq \psi,$$
$$(t_1-t_2)-\mathbf{n}\cdot(\mathbf{r}_1-\mathbf{r}_2) \simeq -t_0\left[\frac{1}{2}(\gamma^{-2}+\delta^2+\psi^2)+\frac{1}{24}\omega_0^2 t_0^2\right]. \tag{17.28}$$

Substituting (17.28) into (17.25), we find the angular spectral distribution of the radiated power of the σ- and the π-component:

$$\begin{pmatrix} dW_\sigma \\ dW_\pi \end{pmatrix} = \frac{e^2\omega^2}{(2\pi)^2}\, d\omega\, d\delta\, d\psi \int_{-\infty}^{\infty} dt_0 \begin{pmatrix} \delta^2-\frac{1}{4}\omega_0^2 t_0^2 \\ \psi^2 \end{pmatrix} \cdot$$
$$\times \exp\left\{-i\omega t_0\left[\frac{1}{2}(\gamma^{-2}+\delta^2+\psi^2)+\frac{1}{24}\omega_0^2 t_0^2\right]\right\}. \tag{17.29}$$

17.3 Angular Spectral Distribution of Radiated Power for Relativistic Particles

Since the radiation propagating in direction $\mathbf{n}$ is gathered from the portion of the electron trajectory on which $\mathbf{v}$ rotates through an angle of the order of γ^{-1} (see (17.20)), it appears interesting to calculate the radiated power (17.29) integrated over the rotation angles δ; the limits of integration with respect to δ can be extended to infinity because the integrand falls off rapidly at large values of δ. If, for the sake of convenience, we introduce new variables x and y instead of t_0 and δ via

$$\omega_0 t_0 = b(x+y), \quad \delta = \frac{b}{2}(x-y),$$
$$b = (2\omega_0/\omega)^{1/3}, \quad |\partial(t_0,\delta)/\partial(x,y)| = b^2/\omega_0, \tag{17.30}$$

we get

$$\begin{pmatrix} dW_\sigma \\ dW_\pi \end{pmatrix} = \frac{e^2\omega^2}{(2\pi)^2}\, d\omega\, d\psi\, \frac{b^2}{\omega_0} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \begin{pmatrix} -b^2xy \\ \psi^2 \end{pmatrix} \times \exp\left\{-i\left(zx + \frac{x^3}{3} + zy + \frac{y^3}{3}\right)\right\}, \tag{17.31}$$

where

$$z = \left(\frac{\omega}{2\gamma^3\omega_0}\right)^{2/3} (1 + \gamma^2\psi^2). \tag{17.32}$$

We see that (17.31) contains integrals of the type

$$f_n(z) = \int_{-\infty}^{\infty} dx x^n \exp\left[i\left(zx + \frac{1}{3}x^3\right)\right]. \tag{17.33}$$

with $n = 0, 1$. Such integrals are expressed in terms of the well-known Airy function Φ, which is defined by the integral representation

$$\Phi(z) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[i\left(zt + \frac{1}{3}t^3\right)\right] dt \tag{17.34}$$

and satisfies the Airy differential equation

$$\Phi''(z) - z\Phi(z) = 0. \tag{17.34a}$$

Using (17.34), we can find the integrals (17.33):

$$\begin{aligned} f_0(z) &= 2\sqrt{\pi}\,\Phi(z), \\ f_1(z) &= -if_0'(z) = -2i\sqrt{\pi}\,\Phi'(z), \\ f_2(z) &= -f_0''(z) = -2\sqrt{\pi}\, z\Phi(z). \end{aligned} \tag{17.35}$$

Allowing for (17.35) and (17.33) and integrating with respect to x and y in (17.31), we arrive at the following formula for the radiated power of synchrotron radiation over frequency ω, angle ψ, and type of polarization $\lambda = \sigma, \pi$:

$$\begin{pmatrix} dW_\sigma \\ dW_\pi \end{pmatrix} = \frac{2}{\pi}\frac{e^2}{\gamma^2}\,\omega\, d\omega\, d\tau \left(\frac{\omega}{2\omega_c}\right)^{1/3} \begin{pmatrix} (2\omega_c/\omega)^{2/3}\Phi'^2(z) \\ \tau^2\Phi^2(z) \end{pmatrix}, \tag{17.36}$$

where instead of angle ψ we have introduced the variable

$$\tau = \gamma\psi, \tag{17.37}$$

and the critical frequency (see (17.23)) is

$$\omega_c = \gamma^3\omega_0, \tag{17.38}$$

so that the argument (17.32) of the Airy function and the derivative $\Phi' = d\Phi/dz$ assumes the form

$$z = \left(\frac{\omega}{2\omega_c}\right)^{2/3}(1+\tau^2). \tag{17.39}$$

Note that formula (17.36) is valid for ultrarelativistic particles. The maximum in the spectrum lies in the frequency range for which the argument z of the Airy function is approximately equal to or less than unity (for $z \gg 1$, as (17.34) clearly demonstrates, $\Phi(z)$ rapidly falls off). Hence, inside the radiation cone ($\tau = \gamma\psi \leqslant 1$) the maximum occurs at the frequency (see (17.39)) $\omega_{max} \sim \omega_c$ given by (17.38), in accord with the estimate (17.23). Therefore, in the case of periodic motion, the discrete spectrum (17.14) becomes practically continuous for $\gamma \gg 1$, since separate higher harmonics $\omega_\nu = \nu\omega_0$ are not resolved any more and the spectrum can be described approximately by formula (17.36). We note, in passing, that (17.36) can be obtained directly from (17.14) via the asymptotic representation of the Bessel functions in terms of the Airy functions or the related MacDonald (modified Hankel) functions (see [2]).

17.4 Spectral Distribution of Synchrotron Radiation

The synchrotron-radiation spectrum can be obtained from (17.36) via integration with respect to τ, but it is simpler to return to formula (17.29) and introduce new variables (the polar coordinates in the $\delta\psi$-plane) instead of δ and ψ:

$$\tau = \gamma\psi = \rho\cos\alpha,\ \gamma\delta = \rho\sin\alpha,\ 0 \leqslant \rho < \infty,$$
$$0 \leqslant \alpha < 2\pi.$$

Integration with respect to α is trivial because the expression inside the square brackets in (17.29) is independent of α. What remains is the double integral with respect to ρ^2 and t_0:

$$\begin{pmatrix} dW_\sigma \\ dW_\pi \end{pmatrix} = \frac{e^2\omega^2}{8\pi\gamma^2}\,d\omega \int_0^\infty d\rho^2 \int_{-\infty}^{\infty} dt_0 \begin{pmatrix} \gamma^{-2}\rho^2 - \frac{1}{2}\omega_0^2 t_0^2 \\ \gamma^{-2}\rho^2 \end{pmatrix}$$
$$\times \exp\left[-i\,\frac{\omega t_0}{2\gamma^2}\left(1+\rho^2+\frac{1}{12}\,\gamma^2\omega_0^2 t_0^2\right)\right],$$

which after the change of variables

$$t' = \frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^{1/3}\omega_0 t_0, \quad u = \left(\frac{\omega}{\gamma^3\omega_0}\right)^{2/3}(1+\rho^2)$$

assumes the form

$$\begin{pmatrix} dW_\sigma \\ dW_\pi \end{pmatrix} = \frac{e^2\omega^2}{4\pi\gamma^2}\, d\omega \int_x^\infty du \int_{-\infty}^\infty dt' \begin{pmatrix} \frac{u}{x} - 1 - \frac{2t'^2}{x} \\ \frac{u}{x} - 1 \end{pmatrix}$$

$$\times \exp\left[-i\left(ut' + \frac{1}{3}t'^3\right)\right].$$

Integration with respect to t' is carried out by formulas (17.33) and (17.35). The remaining integrals with respect to u are transformed thus:

$$\int_x^\infty \Phi(u)\, du \equiv \Phi_1(x),$$

$$\int_x^\infty u\Phi(u)\, du = \int_x^\infty \Phi''(u)\, du = -\Phi'(x).$$

As a result we obtain a spectral distribution of the radiated power for the σ- and the π-component in the form

$$\begin{pmatrix} dW_\sigma \\ dW_\pi \end{pmatrix} = -\frac{e^2\omega}{2\sqrt{\pi}\,\gamma^2}\, d\omega \begin{pmatrix} \Phi_1(x) + \frac{3}{x}\Phi'(x) \\ \Phi_1(x) + \frac{1}{x}\Phi'(x) \end{pmatrix}, \qquad (17.40)$$

where

$$\Phi_1(x) = \int_x^\infty \Phi(t)\, dt, \quad x = \left(\frac{\omega}{\omega_c}\right)^{2/3}, \quad \omega_c = \gamma^3\omega_0. \qquad (17.41)$$

Let us find the total radiated powers for the σ- and the π-component, W_σ and W_π, by integration with respect to ω in (17.40). After we go over to the variable x in (17.41), the calculations amount to evaluating the integrals

$$\int_0^\infty x\Phi'(x)\, dx = -\int_0^\infty \Phi(x)\, dx = -\Phi_1(0),$$

$$\int_0^\infty x^2\Phi_1(x)\, dx = \frac{1}{3}\int_0^\infty x^3\Phi(x)\, dx = -\frac{2}{3}\int_0^\infty x\Phi'(x)\, dx = \frac{2}{3}\Phi_1(0),$$

where we have employed integration by parts and the following relationships (see (17.41) and (17.34a)): $\Phi_1'(x) = -\Phi(x)$ and $x\Phi(x) = \Phi''(x)$. The value of $\Phi_1(0)$ can easily be found by substituting

(17.34) into the definition (17.41) of $\Phi_1(x)$ and proves to be equal to

$$\Phi_1(0) = \frac{\sqrt{\pi}}{3}. \tag{17.41a}$$

As a result we get

$$\begin{pmatrix} W_\sigma \\ W_\pi \end{pmatrix} = W \begin{pmatrix} 7/8 \\ 1/8 \end{pmatrix}. \tag{17.42}$$

Here

$$W = W_\sigma + W_\pi = \frac{2}{3} e^2 \omega_0^2 \gamma^4 = \frac{2}{3} \frac{e^2}{R^2} \left(\frac{\varepsilon}{m}\right)^4 \tag{17.43}$$

is the total radiated power of synchrotron radiation, a formula that can easily be obtained directly from the general formula (16.32) if we allow for (17.2) and (17.27). From (17.42) it follows that synchrotron radiation, on the whole, is strongly polarized linearly in the plane of the electron orbit, with the degree of polarization being

$$p = \frac{W_\sigma - W_\pi}{W_\sigma + W_\pi} = \frac{3}{4}. \tag{17.44}$$

The angular distribution of the radiated power for the σ- and the π-component will be considered in Section 36 (see (36.12) and (36.13)).

Let us consider in greater detail the spectrum of the total radiated power for synchrotron radiation. This spectrum is obtained by summing in (17.40) over the polarization types:

$$\frac{dW}{d\omega} = -\frac{e^2\omega}{\sqrt{\pi}\gamma^2}\left[\Phi_1(x) + \frac{2}{x}\Phi'(x)\right], \quad x = (\omega/\omega_c)^{2/3}. \tag{17.45}$$

Let us study the limiting cases of low and high frequencies. For $\omega \ll \omega_c$ ($x \ll 1$) the main contribution to (17.45) is provided by the term

$$(2/x)\,\Phi'(x) \simeq (2/x)\,\Phi'(0),$$

and with due regard for

$$\Phi'(0) = -\frac{3^{1/6}}{2\sqrt{\pi}}\Gamma(2/3) \tag{17.46}$$

we arrive at the following formula for the low-frequency part of the spectrum:

$$\frac{dW}{d\omega} \simeq \frac{3^{1/6}}{\pi}\Gamma(2/3)\frac{e^2\omega_c}{\gamma^2}\left(\frac{\omega}{\omega_c}\right)^{1/3}. \tag{17.47}$$

Thus, in the low-frequency range the radiated power for synchrotron radiation grows with ω according to a power law: $dW/d\omega \propto \omega^{1/3}$.

For $\omega \gg \omega_c$ ($x \gg 1$) we employ the formulas for the asymptotic behavior of the Airy function, of the derivative of this function, and of the integral (17.41):

$$\begin{aligned} \Phi(z) &\simeq \frac{1}{2} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right), \\ \Phi'(z) &\simeq -\frac{1}{2} z^{1/4} \exp\left(-\frac{2}{3} z^{3/2}\right), \\ \Phi_1(z) &\simeq \frac{1}{2} z^{-3/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \quad \text{as} \quad z \to +\infty. \end{aligned} \tag{17.48}$$

Combining this with (17.45), we arrive at a formula for the spectrum in the high-frequency region ($\omega \gg \omega_c$):

$$\frac{dW}{d\omega} \simeq \frac{1}{2\sqrt{\pi}} \frac{e^2 \omega_c}{\gamma^2} \left(\frac{\omega}{\omega_c}\right)^{1/2} \exp\left(-\frac{2}{3} \frac{\omega}{\omega_c}\right). \tag{17.49}$$

Hence, at high frequencies, the radiated power of synchrotron radiation falls off exponentially. From (17.47) and (17.49) it also follows that the maximum in the radiation spectrum is attained at $\omega \sim \omega_c$ and the spectrum extends (with exponential accuracy) to $\omega \lesssim \omega_c$. This agrees with the qualitative estimates that we made in Section 17.2 (see also the end of Section 17.3).

For practical applications it has proved convenient to express the spectral distribution of synchrotron radiation, (17.45), in terms of the MacDonald functions K_p, with which the Airy function (17.34) and its derivative are related thus:

$$\begin{aligned} \Phi(z) &= \left(\frac{z}{3\pi}\right)^{1/2} K_{1/3}\left(\frac{2}{3} z^{3/2}\right), \\ \Phi'(z) &= -\frac{z}{(3\pi)^{1/2}} K_{2/3}\left(\frac{2}{3} z^{3/2}\right) \quad \text{for} \quad z > 0. \end{aligned} \tag{17.50}$$

If we now allow for (17.50) and the well-known formula

$$2K'_{2/3}(x) + K_{1/3}(x) = -K_{5/3}(x), \tag{17.51}$$

we arrive at a formula for the synchrotron-radiation spectrum in a form convenient for numerical calculations:

$$\frac{dW}{dy} = W f(y), \quad y = \frac{2}{3} \frac{\omega}{\omega_c}, \quad f(y) = \frac{9\sqrt{3}}{8\pi} y \int_y^\infty K_{5/3}(x)\, dx. \tag{17.52}$$

The graph of the universal function $f(y)$ describing the spectral distribution of the radiated power in the case of synchrotron radiation is depicted in Figure 17.1.

Numerical calculations show that the maximum in the spectrum (function $f(y)$ in (17.52)) is attained at $y \simeq 1/3$, which corresponds to a frequency

$$\omega_{max} \simeq \frac{1}{2}\omega_c = \frac{1}{2}\gamma^3\omega_0 \tag{17.53}$$

or a wavelength

$$\lambda_{max} = \frac{2\pi c}{\omega_{max}} \simeq 4\pi R\left(\frac{mc^2}{\varepsilon}\right)^3$$

(we have used the common system of units).

Thus, as the electron energy ε is increased, the maximum in the synchrotron-radiation spectrum moves into the region of shorter waves. Let us estimate λ_{max} for parameters of real accelerators. At

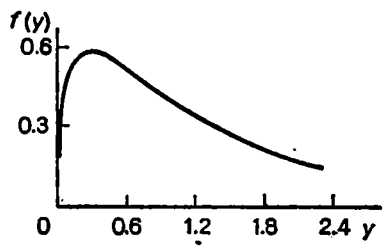

Figure 17.1 The universal curve describing the spectral distribution of the radiated power of synchrotron radiation.

$\varepsilon = 50$ MeV ($\gamma = 10^2$) and an orbit radius $R = 1$ m we get $\lambda_{max} \sim 10^3$ Å, which corresponds to the visible part of the spectrum (a "luminous electron"). At $\varepsilon = 500$ MeV ($\gamma = 10^3$) and $R \sim 1$ m we get $\lambda_{max} \sim 10$ Å (far-ultraviolet radiation). Finally, for a synchrotron with $\varepsilon = 5$ GeV ($\gamma = 10^4$) and $R \sim 1$ m we have $\lambda_{max} \sim 0.1$ Å (X rays)

The unique properties of synchrotron radiation (sharp angular directivity, strong linear polarization, and a broad spectrum with a maximum in the high-frequency range) brought about the wide use of this radiation in physical experiments and in other fields of science and technology (see Section 36).

The quantum effects inherent in synchrotron radiation, which manifest themselves at high electron energies and high magnetic fields, will be considered in Section 29.

Chapter III

The Basics of Quantum Electrodynamics

18. Quantization of the Free Electromagnetic Field (Coulomb Gauge)

The electromagnetic field constitutes a continuous system with an infinite number of degrees of freedom. This puts it apart from discrete systems with a finite number of degrees of freedom. The quantization of discrete systems, which is carried out in quantum mechanics, consists in imposing certain commutation relations on the operators that represent physical quantities. These operators must be defined in an appropriate way before quantization is enforced. The common approach to quantization is to take the following commutation relation between the momentum p_j and coordinate q_i operators:

$$[q_i, p_j] = i\hbar\delta_{ij} \tag{18.1}$$

where i and j label the degrees of freedom corresponding to the coordinate and the momentum.

However, (18.1) cannot be used to describe fields, since the variables in the field functions $u = u(\mathbf{x})$ depend on the "number" of the particular degree of freedom, that is, coordinate $\mathbf{x}$, in a continuous manner. Two approaches exist for overcoming this difficulty. One consists in partitioning the three-dimensional space in which the field is "moving" into discrete cells, imposing a periodicity condition on the field functions, with the period encompassing a sufficiently large (but finite) number of cells, and quantizing in a standard manner the resulting discrete system with a finite number of degrees of freedom over a single period.

The other approach consists in going over from the coordinate representation to the momentum representation, in which picture the periodicity condition results in the momentum values becoming discrete. The standard quantization procedure can then be applied to such a system. Here we use the second approach since it is more pictorial than the first and reduces the problem of field quantization to the well-known problem of a system of harmonic oscillators.

18.1 The Energy of the Free Electromagnetic Field. Photons

In the Coulomb gauge,

$$A^0 = 0, \quad \text{div } \mathbf{A} = 0, \tag{18.2}$$

the vector potential of the free electromagnetic field, $\mathbf{A}(\mathbf{r}, t)$, obeys d'Alembert's equation (see Sections 10, 12, and 13),

$$\Box \mathbf{A}(\mathbf{r}, t) = 0. \tag{18.3}$$

The real-valued general solution of this wave equation can be represented by the following Fourier series:

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}} \sqrt{\frac{2\pi}{\omega}} \left(\mathbf{a}_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{a}_{\mathbf{k}}^*(t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\right). \tag{18.4}$$

Here L is the length of the principal periodicity cube (the three-dimensional space has been partitioned into such cubes), the wave vector $\mathbf{k}$ assumes discrete values with components

$$k_x = 2\pi n_x/L, \quad k_y = 2\pi n_y/L, \quad k_z = 2\pi n_z/L, \tag{18.5}$$

$n_x, n_y, n_z = 0, \pm 1, \pm 2, \ldots$, and the meaning of the factor $\sqrt{2\pi/\omega}$ will become clearer from what follows.

To satisfy d'Alembert's equation (18.3), the Fourier coefficients must obey the following equations:

$$\frac{d^2\mathbf{a}_{\mathbf{k}}}{dt^2} + \mathbf{k}^2 \cdot \mathbf{a}_{\mathbf{k}} = 0, \quad \frac{d^2\mathbf{a}_{\mathbf{k}}^*}{dt^2} + \mathbf{k}^2 \cdot \mathbf{a}_{\mathbf{k}}^* = 0. \tag{18.6}$$

The fact that potential (18.4) is real implies that the solutions to Eqs. (18.6) are complex-valued, namely,

$$\mathbf{a}_{\mathbf{k}}(t) = \mathbf{a}_{\mathbf{k}}(0)\, e^{-i\omega t}, \quad \omega = |\mathbf{k}|, \tag{18.7}$$

and

$$\mathbf{a}_{\mathbf{k}}^*(t) = \mathbf{a}_{\mathbf{k}}^*(0)\, e^{i\omega t} = \mathbf{a}_{-\mathbf{k}}(0) e^{i\omega t}. \tag{18.8}$$

The Coulomb gauge (18.2) forces us to conclude that monochromatic components of the field are transverse, that is,

$$\mathbf{k} \cdot \mathbf{a}_{\mathbf{k}} = 0. \tag{18.9}$$

Therefore, in the plane orthogonal to vector $\mathbf{k}$ we can select two base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$,

$$\mathbf{e}_1 \perp \mathbf{e}_2, \quad \mathbf{e}_1 \perp \mathbf{k}, \quad \mathbf{e}_2 \perp \mathbf{k}, \tag{18.10}$$

and represent $\mathbf{a}_{\mathbf{k}}$ in the form

$$\mathbf{a}_{\mathbf{k}} = a_{\mathbf{k}_1}\mathbf{e}_1 + a_{\mathbf{k}_2}\mathbf{e}_2 = \mathbf{a}_{\mathbf{k}_s}\mathbf{e}_s. \tag{18.11}$$

The energy of the electromagnetic field,

$$H = \int d^3x \mathscr{H}, \quad \mathscr{H} = \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi}, \tag{18.12}$$

can be expressed in terms of the Fourier coefficients if we recall that $\mathbf{E} = -\mathbf{A}_{,0}$ and $\mathbf{H} = \operatorname{curl} \mathbf{A}$ and use the expansion (18.4). The integral over the volume yields Kronecker deltas,

$$\frac{1}{L^3} \int d^3x e^{\pm i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = \delta_{\mathbf{k}\mathbf{k}'} \quad (\delta_{\mathbf{k}\mathbf{k}'} \equiv \delta_{n_x n_x'} \delta_{n_y n_y'} \delta_{n_z n_z'}), \tag{18.13}$$

while the transversality condition (18.9) leads to

$$(\mathbf{k} \times \mathbf{a}_{\mathbf{k}}) \cdot (\mathbf{k} \times \mathbf{a}_{\mathbf{k}}^*) = \omega^2 \mathbf{a}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}^*. \tag{18.14}$$

The result is

$$H = \sum_{\mathbf{k},\, s=1,2} \frac{\omega}{2} (a_{\mathbf{k}s} a_{\mathbf{k}s}^* + a_{\mathbf{k}s}^* a_{\mathbf{k}s}). \tag{18.15}$$

As we can see, the dynamical description of the electromagnetic field has reduced itself to the problem of describing an infinite discrete set of dynamical systems, with each system described by the oscillation equations (18.6). Since only complex-valued solutions (18.7) and (18.8) are allowed, the quantization procedure requires imposing on the operators $\mathbf{a}_{\mathbf{k}s}$ and $\mathbf{a}_{\mathbf{k}s}^+$, which correspond to the Fourier components $a_{\mathbf{k}s}$ and $a_{\mathbf{k}s}^*$, the following conditions:

$$[\mathbf{a}_{\mathbf{k}s}, \mathbf{a}_{\mathbf{k}'s'}^+] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'}. \tag{18.16}$$

The real combinations

$$q_{\mathbf{k}s} = \frac{1}{\sqrt{2}} (a_{\mathbf{k}s} + a_{\mathbf{k}s}^*), \qquad p_{\mathbf{k}s} = -\frac{i}{\sqrt{2}} (a_{\mathbf{k}s} - a_{\mathbf{k}s}^*) \tag{18.17}$$

have corresponding to them the operators $\mathbf{q}_{\mathbf{k}s}$ and $\mathbf{p}_{\mathbf{k}s}$ satisfying the common commutation relations (18.1) for the coordinate and momentum:

$$[\mathbf{q}_{\mathbf{k}s}, \mathbf{p}_{\mathbf{k}'s'}] = i\delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'}. \tag{18.18}$$

As is known, the quantization of an oscillator with the Hamiltonian

$$\mathbf{H}_{\mathbf{k}s} = \frac{\omega}{2} (\mathbf{a}_{\mathbf{k}s} \mathbf{a}_{\mathbf{k}s}^+ + \mathbf{a}_{\mathbf{k}s}^+ \mathbf{a}_{\mathbf{k}s}), \tag{18.19}$$

under condition (18.6) leads to the following eigenvalues of the energy

$$E(N_{\mathbf{k}s}) = \omega(N_{\mathbf{k}s} + 1/2), \tag{18.20}$$

where $N_{\mathbf{k}s} = 0, 1, 2, \ldots$. The operators $a_{\mathbf{k}s}$ and $a^+_{\mathbf{k}s}$ are represented by the following matrices with an infinite number of rows and columns:

$$a_{\mathbf{k}s} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & . \\ 0 & 0 & \sqrt{2} & 0 & . \\ 0 & 0 & 0 & \sqrt{3} & . \\ . & . & . & . & . \end{pmatrix}, \quad a^+_{\mathbf{k}s} = \begin{pmatrix} 0 & 0 & 0 & . \\ \sqrt{1} & 0 & 0 & . \\ 0 & \sqrt{2} & 0 & . \\ 0 & 0 & \sqrt{3} & . \\ . & . & . & . \end{pmatrix}, \tag{18.21}$$

that is, the only nonzero matrix elements are

$$\langle N_{\mathbf{k}s} - 1 \mid a_{\mathbf{k}s} \mid N_{\mathbf{k}s} \rangle = \langle N_{\mathbf{k}s} \mid a^+_{\mathbf{k}s} \mid N_{\mathbf{k}s} - 1 \rangle = \sqrt{N_{\mathbf{k}s}}. \tag{18.22}$$

For a system of oscillators with the Hamiltonian

$$H = \sum_{\mathbf{k},\, s=1,\, 2} \frac{\omega}{2} (a_{\mathbf{k}s} a^+_{\mathbf{k}s} + a^+_{\mathbf{k}s} a_{\mathbf{k}s}) \tag{18.23}$$

we obtain the following eigenvalues of the energy:

$$E = \sum_{\mathbf{k},\, s=1,\, 2} \omega (N_{\mathbf{k}s} + 1/2), \tag{18.24}$$

corresponding to various sets of nonnegative integers $N_{\mathbf{k}s}$. Formula (18.24), which is simply the sum of energies of quantum oscillators, or the so-called modes characterized by $\mathbf{k}$ and s, can be interpreted differently. The energy quanta corresponding to the oscillators ω can be interpreted as the energies of particles, or photons, while the numbers $N_{\mathbf{k}s} = 0, 1, 2, \ldots$ can be interpreted as the numbers of such photons with definite values of momentum $\mathbf{k}$ and polarization s, that is, the occupation numbers of the states $\mathbf{k}$, s. Then (18.24) stands for the total energy of the quantized electromagnetic field in a state with a definite number of photons, $N_{\mathbf{k}s}$. The infinitely large positive (constant) quantity corresponding to the ground state of the system, $N_{\mathbf{k}s} = 0$, can be removed by redefining the Hamiltonian H (see below).

The ground, or vacuum, state of the electromagnetic field $|0\rangle$ satisfies the equation

$$a_{\mathbf{k}s} |0\rangle = 0 \tag{18.25}$$

for all $\mathbf{k}$ and s. The derivation of this condition is known from the theory of a quantum harmonic oscillator. The excited states with a

definite number of photons $N_{\mathbf{k}s}$ are obtained, via a well-known procedure, from the vacuum state. The result is

$$|N_{\mathbf{k}s}\rangle = \frac{(a^+_{\mathbf{k}s})^{N_{\mathbf{k}s}}}{\sqrt{N_{\mathbf{k}s}!}}|0\rangle. \tag{18.26}$$

These states are the eigenstates of the occupation number operator

$$N_{\mathbf{k}s} = a^+_{\mathbf{k}s}a_{\mathbf{k}s}. \tag{18.27}$$

Of course, vector (18.26) describes only a single mode $\mathbf{k}$, s. The total state vector of the field is obtained by acting with operators $a^+_{\mathbf{k}s}$ of each mode raised to the corresponding powers (corresponding to the occupation numbers $N_{\mathbf{k}s}$) on the vacuum state $|0\rangle$. The expression for the momentum of the electromagnetic field follows from the general definition (4.29). If we also allow for (10.30), we get

$$\mathbf{P} = \int d^3x \frac{\partial \mathcal{L}}{\partial A_{\mu,t}} \nabla A_\mu = \frac{1}{4\pi}\int \mathbf{E}\times\mathbf{H}d^3x. \tag{18.28}$$

In the Fourier representation, (18.28) yields

$$\mathbf{P} = \sum_{\mathbf{k},\, s=1,2} \mathbf{k}\,(a^*_{\mathbf{k}s}a_{\mathbf{k}s} + a_{\mathbf{k}s}a^*_{\mathbf{k}s}). \tag{18.29}$$

Going over to operators, we get

$$\mathbf{P} = \sum_{\mathbf{k},\, s=1,2} \mathbf{k}\,(a^+_{\mathbf{k}s}a_{\mathbf{k}s} + a_{\mathbf{k}s}a^+_{\mathbf{k}s}), \tag{18.30}$$

which is the operator of the electromagnetic-field momentum. Its eigenvalues are

$$\mathbf{P} = \sum_{\mathbf{k},\, s=1,2} \mathbf{k}\,(N_{\mathbf{k}s} + 1/2). \tag{18.31}$$

This result corroborates the above interpretation of $\mathbf{k}$ as the photon momentum. Recalling the relationship $\omega = |\mathbf{k}|$, which links the photon energy to the photon momentum, we conclude that the photon is massless, which is in complete agreement with the fact that the electromagnetic field, being a vector gauge field, is massless (see Section 10).

Now let us investigate the question of defining the energy of the electromagnetic field. First we note that the energy of the classical field, (18.15), is positive definite and, obviously, does not depend on the order in which the factors $a_{\mathbf{k}s}$ and $a^*_{\mathbf{k}s}$ follow in (18.15). The quantum definition of the Hamiltonian operator (18.23) copies the classical expression (18.15), with the classical Fourier coefficients

replaced with the operators a_{ks} and a^+_{ks}. The commutation relation (18.16) makes it possible to write (18.23) in the form

$$H = \sum_{k,\, s=1,\, 2} \omega \left(a^+_{ks} a_{ks} + \frac{1}{2} \right), \tag{18.32}$$

where $a^+_{ks} a_{ks} = N_{ks}$ is the occupation number operator, which has nonnegative integral eigenvalues. Thus, the energy of the quantized electromagnetic field assumes nonnegative values, just as the energy of the classical electromagnetic field does. However, if we define operator H in such a manner, the energy acquires an infinitely large positive term

$$E_0 = \sum_{k,\, s=1,\, 2} \frac{1}{2} \omega, \tag{18.33}$$

which corresponds to the ground state of the field, or the vacuum state. This term, which has no physical meaning, can be eliminated and thus the energy of the vacuum state can be made equal to zero, $E_0 = 0$, if we define H differently. Indeed, the definition of the Hamiltonian operator (18.23), which copies (18.15), can be replaced with another definition in which the order in which a_{ks} and a^+_{ks} operate is changed. If we start with the classical definition (18.15), the Fourier coefficients can be freely interchanged, say, to obtain

$$H = \sum_{k,\, s=1,\, 2} \omega a^*_{ks} a_{ks}. \tag{18.34}$$

In accord with this formula, instead of (18.23) we can introduce an alternative definition:

$$H = \sum_{k,\, s=1,\, 2} \omega a^+_{ks} a_{ks}. \tag{18.35}$$

Then the infinitely large term in the field's energy does not appear, the energy of the vacuum state, E_0, is zero, and the energy of an excited state of the field is

$$E = \sum_{k,\, s=1,\, 2} N_{ks} \omega. \tag{18.36}$$

Definition (18.35) corresponds to the so-called normal order in which the operators act (the normal product, see below), that is, an order in which the creation operators a^+_{ks} stand to the left of the annihilation operators a_{ks} in a product.

Thus, quantization of the electromagnetic field results in a replacement of the classical expansion of the vector potential (18.4) with the quantum operator

$$A(r) = \frac{1}{L^{3/2}} \sum_{k,\, s=1,\, 2} \sqrt{\frac{2\pi}{\omega}} \left(e_s a_{ks} e^{ik \cdot r} + e^*_s a^+_{ks} e^{-ik \cdot r} \right), \tag{18.37}$$

where the $\mathbf{e}_s$ are the vectors of transverse polarization ($s = 1, 2$), and $a^+_{\mathbf{k}s}$ and $a_{\mathbf{k}s}$ are the photon creation and annihilation operators (these operators obey the commutation relations (18.16)). The operators $a_{\mathbf{k}s}$ and $a^+_{\mathbf{k}s}$ and the vector potential operator $\mathbf{A}(\mathbf{r})$ do not depend on time explicitly (in contrast to the classical case where, say, $\mathbf{A} = \mathbf{A}(\mathbf{r}, t)$), since they are given in the Schrödinger representation. Time dependence of the operators is achieved if we go over to the Heisenberg representation (see Section 19).

18.2 Photon Spin

In accord with the general definition of the spin of a vector field, (9.39), the electromagnetic-field spin tensor is

$$S_{ij} = \int d^3x\,(u_i\pi_j - u_j\pi_i), \tag{18.38}$$

where $u_i = A_i$ and $\pi_j = -E_j/4\pi$. Then the electromagnetic-field spin vector proves to be

$$\mathbf{S} = \frac{1}{4\pi}\int d^3x\,\mathbf{E}\times\mathbf{A}. \tag{18.39}$$

Substituting the expansion (18.37), we get the spin operator

$$\mathbf{S} = \sum_{\mathbf{k}} \frac{i}{2}(\mathbf{a}_{\mathbf{k}}\times\mathbf{a}^+_{\mathbf{k}} - \mathbf{a}^+_{\mathbf{k}}\times\mathbf{a}_{\mathbf{k}}), \tag{18.40}$$

where $\mathbf{a}_{\mathbf{k}} = \sum_s a_{\mathbf{k}s}\mathbf{e}_s$, or in explicit form,

$$\mathbf{S} = -i\sum_{\mathbf{k}}(a^+_{\mathbf{k}1}a_{\mathbf{k}2} - a^+_{\mathbf{k}2}a_{\mathbf{k}1})\,\mathbf{n}. \tag{18.41}$$

We introduce complex combinations of the creation and annihilation operators,

$$c_{\mathbf{k}}(1) = \frac{a_{\mathbf{k}1} - ia_{\mathbf{k}2}}{\sqrt{2}}, \quad c_{\mathbf{k}}(-1) = \frac{a_{\mathbf{k}1} + ia_{\mathbf{k}2}}{\sqrt{2}}, \tag{18.42}$$

in terms of which the spin operator is expressed in a diagonal manner:

$$\mathbf{S} = \sum_{\mathbf{k}}(c^+_{\mathbf{k}}(1)\,c_{\mathbf{k}}(1) - c^+_{\mathbf{k}}(-1)\,c_{\mathbf{k}}(-1))\,\mathbf{n}, \tag{18.43}$$

where $\mathbf{n} = \mathbf{k}/\omega$ is a unit vector, $|\mathbf{n}| = 1$.

The new operators $c_{\mathbf{k}}(i)$ and $c^+_{\mathbf{k}}(i)$ defined in (18.42) obey the same commutation relations (18.16) as the operators $a_{\mathbf{k}s}$ and $a^+_{\mathbf{k}s}$, and the Hamiltonian operator H has the same form (18.35):

$$H = \sum_{\mathbf{k},\, i=+1,\,-1} c^+_{\mathbf{k}}(i)\,c_{\mathbf{k}}(i)\,\omega. \tag{18.44}$$

Thus, the new operators have the same matrix elements (18.22) and the operator $c_{\mathbf{k}}^{\dagger}(i)\, c_{\mathbf{k}}(i)$ possesses nonnegative integral eigenvalues and can be identified with the photon occupation number operator. The diagonal spin operator, (18.43), possesses the following eigenvalues:

$$\mathbf{S} = \sum_{\mathbf{k}} (N_{\mathbf{k}}(1) - N_{\mathbf{k}}(-1))\,\mathbf{n}. \tag{18.45}$$

It is now clear that $N_{\mathbf{k}}(1)$ is the number of photons with spin 1 directed along $\mathbf{n}$, while $N_{\mathbf{k}}(-1)$ is the number of photons with spin 1 directed along $-\mathbf{n}$.

In accordance with the fact that the electromagnetic field is transverse, vector $\mathbf{A}$ has only two linear polarizations $s = 1, 2$ (from which two circular polarizations are constructed), one parallel to the direction of wave propagation and the other opposite to that direction. Therefore, the operators $c_{\mathbf{k}}^{\dagger}(1)$ and $c_{\mathbf{k}}^{\dagger}(-1)$ correspond to the creation of photons with right-handed circular polarization (spin directed along $\mathbf{n}$) and left-handed circular polarization (spin directed along $-\mathbf{n}$), respectively. The expression (18.45) shows also that the spin of the electromagnetic field is unity, just as of any vector field. Moreover, quantization combined with the commutation relations (18.16) leads to arbitrary values of the occupation numbers $N_{\mathbf{k}s} = 0, 1, 2, 3, \ldots$, a situation characteristic of the Bose-Einstein statistics. This fact demonstrates that there is a connection between spin and statistics, with the result that vector fields are quantized with due regard for the Bose statistics, or that vector particles are bosons.

It can also be shown that for integral-spin particles (bosons) the quantization rules are fixed by the commutation relations of the (18.16) type; hence, states are possible in which there is an arbitrary number of particles with the same quantum numbers (the Bose-Einstein statistics). For particles with half-integral spin (fermions) the quantization rules obey anticommutation relations (see below); hence, in this case the occupation numbers cannot exceed unity, in accord with Pauli's exclusion principle. This fundamental conclusion concerning the relationship between spin and statistics is known as Pauli's theorem.

19. Quantization of the Electromagnetic Field (Lorentz Gauge)

Quantization of the three-dimensional transverse electromagnetic field in Coulomb gauge, which was carried out in Section 18, is convenient in the study of nonrelativistic processes and in problems involving electromagnetic radiation. More convenient, from the

standpoint of building a general quantum theory of electromagnetic interaction, is the approach that uses Lorentz gauge, since this gauge is Lorentz-invariant.

19.1 The Energy of the Electromagnetic Field

In the general case, neither the temporal component A^0 nor the longitudinal component A^3 of the 4-potential A^μ is zero. For the free field we can write four d'Alembert's equations

$$\Box A_\mu(x) = 0, \tag{19.1}$$

which are obtained from the Lagrange-Euler equations with the Lagrangian density (see Section 10)

$$\mathscr{L} = -\frac{1}{8\pi} A_{\mu,\nu} A^{\mu,\nu}. \tag{19.2}$$

However, Eqs. (19.1) are not equivalent to the Maxwell equations. To obtain the Maxwell equations, it is required in the classical theory that the Lorentz condition be satisfied (see Section 10), or

$$\partial A^\mu / \partial x^\mu = 0. \tag{19.3}$$

The general solution to d'Alembert's equation can be represented by a Fourier series for the 4-potential A_μ, similar to the three-dimensional case:

$$A_\mu(x) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}\lambda} \sqrt{\frac{2\pi}{\omega}}\, e_\mu^{(\lambda)} (c_{\mathbf{k}\lambda} e^{-ikx} + c^*_{\mathbf{k}\lambda} e^{ikx}), \tag{19.4}$$

where $kx = \omega t - \mathbf{k}\cdot\mathbf{r}$, $\omega = |\mathbf{k}|$, while the polarization unit vectors $e^{(\lambda)}$, or explicitly,

$$\begin{aligned} e^{(0)} &= (1, 0, 0, 0), \quad e^{(1)} = (0, 1, 0, 0), \\ e^{(2)} &= (0, 0, 1, 0), \quad e^{(3)} = (0, 0, 0, 1), \end{aligned} \tag{19.5}$$

possess the following properties:

$$\begin{aligned} e_\mu^{(\lambda)} e^{(\lambda')\mu} = e_0^{(\lambda)} e_0^{(\lambda')} - \mathbf{e}^{(\lambda)}\cdot\mathbf{e}^{(\lambda')} &= \begin{cases} 0 & \text{if } \lambda \neq \lambda', \\ 1 & \text{if } \lambda = \lambda' = 0, \\ -1 & \text{if } \lambda = \lambda' = 1, 2, 3; \end{cases} \\ e_\mu^{(\lambda)} k^\mu &= \begin{cases} k^0 & \text{if } \lambda = 0, \\ -k^0 & \text{if } \lambda = 3, \\ 0 & \text{if } \lambda = 1, 2. \end{cases} \end{aligned} \tag{19.6}$$

The 4-potential (19.4) can be represented as the sum of two terms,

$$A_\mu(x) = A_\mu^{(+)}(x) + A_\mu^{(-)}(x), \tag{19.7}$$

where $A_\mu^{(+)}$ is the positive-frequency part, which consists of terms proportional to exp $(-ikx)$, and $A_\mu^{(-)}$ is the negative-frequency part, which consists of terms proportional to exp (ikx). These two parts differ in the sign in front of frequency ω in the exponents:

$$A_\mu^{(+)} = \sum_{\mathbf{k}} A_{\mu\mathbf{k}}^{(+)} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}, \quad A_\mu^{(-)} = \sum_{\mathbf{k}} A_{\mu\mathbf{k}}^{(-)} e^{+i\omega t + i\mathbf{k}\cdot\mathbf{r}},$$

where in the sum for $A_\mu^{(-)}$ we have replaced $\mathbf{k}$ with $-\mathbf{k}$. As is known (see Section 2), the decomposition (19.7) according to the sign of the temporal component of an isotropic 4-vector, with $k^2 = \omega^2 - \mathbf{k}^2 = 0$, is Lorentz-invariant.

Now let us find the energy of the field,

$$H = \int d^3x \mathscr{H}, \quad \mathscr{H} = A_{\mu,\,t} \frac{\partial \mathscr{L}}{\partial A_{\mu,\,t}} - \mathscr{L} = -\frac{1}{8\pi}[A_{\mu,\,0} A^{\mu,\,0} + (\nabla A_\mu)^2]. \tag{19.8}$$

Allowing for (19.4), we get

$$H = -\sum_{\mathbf{k}} \frac{1}{2}\omega(c_{\mathbf{k}0}^* c_{\mathbf{k}0} + c_{\mathbf{k}0} c_{\mathbf{k}0}^*) + \sum_{\mathbf{k},\,s=1,\,2,\,3} \frac{1}{2}\omega(c_{\mathbf{k}s}^* c_{\mathbf{k}s} + c_{\mathbf{k}s} c_{\mathbf{k}s}^*). \tag{19.9}$$

We see that the energy of the field is not positive definite, as noted earlier in Section 10. To make it positive definite, we must impose the Lorentz condition on the 4-potentials (see Section 10).

19.2 Canonical Quantization

Let us now turn to the problem of quantizing the electromagnetic field. In contrast to the case of Coulomb gauge, we will not use the analogy with quantum oscillators. Instead we employ the method of canonical quantization, consisting in the replacement of the classical Poisson brackets $[H, u]_{\mathrm{cl}}$ with the quantum Poisson brackets i [H, u] in the expression for the time derivative of a dynamical variable u,

$$\frac{du}{dt} = [H, u]_{\mathrm{cl}}, \tag{19.10}$$

where [H, u] is the commutator involving the Hamiltonian operator H and the operator u representing the dynamical variable, or

$$\frac{du}{dt} = i\,[\mathrm{H}, \mathrm{u}]. \tag{19.11}$$

Here the operator u is given in the Heisenberg representation, that is, is time dependent, while the state vectors do not depend on time.

Let us write the quantum Heisenberg operator of the electromagnetic-field 4-potential in the form of a Fourier series:

$$A_\mu(\mathbf{r}, t) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}\lambda} \sqrt{\frac{2\pi}{\omega}}\, e_\mu^{(\lambda)} (c_{\mathbf{k}\lambda}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} + c_{\mathbf{k}\lambda}^+(t)\, e^{-i\mathbf{k}\cdot\mathbf{r}}), \quad (19.12)$$

where $c_{\mathbf{k}\lambda}(t)$ is the Heisenberg (i.e. time dependent) operator representing a Fourier coefficient.

For the operator representing the energy of the field we employ the definition that corresponds to the classical expression (19.9),

$$H = -\frac{1}{2} \sum_{\mathbf{k}} \omega (c_{\mathbf{k}0}^+ c_{\mathbf{k}0} + c_{\mathbf{k}0} c_{\mathbf{k}0}^+) + \frac{1}{2} \sum_{\mathbf{k},\, s=1,\, 2,\, 3} \omega (c_{\mathbf{k}s}^+ c_{\mathbf{k}s} + c_{\mathbf{k}s} c_{\mathbf{k}s}^+). \quad (19.13)$$

Then the dynamical variable $u = c_{\mathbf{k}\lambda}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}}$ will satisfy Eq. (19.11). Substituting expansion (19.13) into this equation, we get the following expression for $u = c_{\mathbf{k}s}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}}$ ($s = 1, 2, 3$):

$$\frac{du}{dt} = \frac{i\omega}{2} [u^+ u + u u^+, u]. \quad (19.14)$$

Note that out of the entire sum (19.13) only the term with quantum numbers $\mathbf{k}$, s that coincide with those of operator u is not commutative with u.

We now use an identity that is valid for any three operators A, B and C:

$$[AB, C] = A[B, C] + [A, C] B.$$

Then instead of (19.14) we get

$$\frac{du}{dt} = -\frac{i\omega}{2} ([u, u^+] u + u [u, u^+]). \quad (19.15)$$

To linearize this equation we must require that the following quantization rules hold true:

$$[u, u^+] = 1. \quad (19.16)$$

Having in mind the remark made after formula (19.14), we can write in general form

$$[c_{\mathbf{k}s}(t), c_{\mathbf{k}'s'}^+(t)] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{ss'}. \quad (19.17)$$

Then instead of (19.15) we obtain the equation

$$\frac{du}{dt} = -i\omega u, \quad (19.18)$$

whose general solution is

$$u = c_{\mathbf{k}s}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}} = c_{\mathbf{k}s} e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}, \quad (19.19)$$

where $c_{\mathbf{k}s}$ is a constant operator coefficient.

In this case the field operator $A_\mu(\mathbf{r}, t)$ proves to satisfy homogeneous d'Alembert's equation (19.1), just as the classical 4-potential does, with (19.19) corresponding to the positive-frequency part of this operator. As for temporal polarization, that is, $u = c_{k0}(t)\, e^{i\mathbf{k}\cdot\mathbf{r}}$, we arrive at equation for it that differs from (19.15) in that the equation contains a "plus" sign in front of the right-hand side. This means that to obtain a positive-frequency solution of the (19.19) type we must fix quantization rules that have a sign opposite to that in (19.16), that is,

$$[u,\ u^+] = -1, \tag{19.20}$$

or, in general form,

$$[c_{k0}(t),\ c^+_{k0}(t)] = -\delta_{kk'}. \tag{19.21}$$

Let us define the vacuum state of the electromagnetic field, $|0\rangle$, by the equation

$$c_{k\lambda}\,|0\rangle = 0, \tag{19.22}$$

where $\lambda = 0, 1, 2, 3$, and $\mathbf{k}$ assumes any value. By analogy with the Coulomb case, we preserve the meaning of $c_{k\lambda}$ and $c^+_{k\lambda}$ as the operators of annihilation and creation of particles (photons):

$$c^+_{k\lambda}\,|0\rangle = |1_{k\lambda}\rangle, \tag{19.23}$$

where $|1_{k\lambda}\rangle$ is the state of a single photon with momentum $\mathbf{k}$ and polarization λ. Then, if we assume that the norm of the vacuum state vector is positive definite, that is,

$$\langle 0\,|\,0\rangle = 1, \tag{19.24}$$

we are forced to conclude that the single-particle state vector of the temporal polarization has a negative norm,

$$\langle 1_{k0}|1_{k0}\rangle = \langle 0|c_{k0}c^+_{k0}|0\rangle = \langle 0|-1+c^+_{k0}c_{k0}|0\rangle = -1, \tag{19.25}$$

as a result of the commutation relations (19.21). Thus, generally, the metric in the space of state vectors $|N_{k\lambda}\rangle$ proves to be indefinite. If we define the state with $N_{k\lambda}$ photons as

$$|N_{k\lambda}\rangle = \frac{(c^+_{k\lambda})^{N_{k\lambda}}}{\sqrt{N_{k\lambda}!}}\,|0\rangle, \tag{19.26}$$

then because of the commutation relations (19.21) we obtain

$$\begin{aligned} \langle N_{ks}|N_{ks}\rangle &= 1, \\ \langle N_{k0}|N_{k0}\rangle &= (-1)^{N_{k0}}. \end{aligned} \tag{19.27}$$

Note that the states $|N_{k0}\rangle$ are eigenvectors of the operator $c^+_{k0}c_{k0}$:

$$c^+_{k0}c_{k0}|N_{k0}\rangle = -N_{k0}|N_{k0}\rangle. \tag{19.28}$$

Indeed, if we denote, for the sake of brevity, c_{k0} by c, then for $N_{k0} = 1$ we have

$$c^+c\,|\,1\rangle = c^+cc^+\,|\,0\rangle = c^+(c^+c-1)\,|\,0\rangle = -c^+\,|\,0\rangle = -|\,1\rangle.$$

If we assume that (19.28) holds true for $|\,N_{k0}-1\rangle$, for $|\,N_{k0}\rangle$ we get

$$c^+c|N\rangle = c^+c\frac{c^+}{\sqrt{N}}\,|N-1\rangle = \frac{1}{\sqrt{N}}c^+(c^+c-1)\,|N-1\rangle$$
$$= (-1+1-N)\,|N\rangle = (-N)\,|N\rangle.$$

We have thus proved that (19.28) is valid in the general case.

In the same manner we can show that the operators c and c^+ act on the state vectors as follows:

$$\begin{aligned} c_{ks}|N_{ks}\rangle &= \sqrt{N_{ks}}\,|N_{ks}-1\rangle, \\ c^+_{ks}|N_{ks}\rangle &= \sqrt{N_{ks}+1}\,|N_{ks}+1\rangle, \\ c_{k0}|N_{k0}\rangle &= -\sqrt{N_{k0}}\,|N_{k0}-1\rangle, \\ c^+_{k0}|N_{k0}\rangle &= \sqrt{N_{k0}+1}\,|N_{k0}+1\rangle, \end{aligned} \tag{19.29}$$

from which, incidentally, (19.28) follows directly. Thus, the operators $c^+_{k\lambda}$ and $c_{k\lambda}$ are indeed the photon creation and annihilation operators. Note that up till now we have imposed no additional conditions on the field operators A_μ and the state vectors. For this reason the states with temporal photons, $|\,\ldots N_{k0}\ldots\rangle$, where the dots stand for the occupation numbers of states with other possible quantum numbers, can provide, due to the indefinite metric (19.27), a negative contribution to the total energy of the electromagnetic field

$$E = \langle H\rangle.$$

For this reason the energy of the field is not positive definite. For E to be nonnegative, we must ensure that the admissible states $|\,\Phi\rangle$ of the field obey the condition

$$\left(\frac{\partial A_\mu}{\partial x_\mu}\right)^{(+)}|\Phi\rangle = 0, \tag{19.30}$$

where by $(\partial A^\mu/\partial x^\mu)^{(+)}$ we denote the positive-frequency part of the divergence $\partial A^\mu/\partial x^\mu$. For the negative-frequency part we obtain, as a corollary to (19.30),

$$\langle\Phi|\left(\frac{\partial A^\mu}{\partial x^\mu}\right)^{(-)} = 0. \tag{19.31}$$

Using expansion (19.12) and solution (19.19), for each value of **k** we arrive at the following gauge condition:

$$\sum_\lambda e^{(\lambda)}_\mu k^\mu c_{k\lambda}|\Phi\rangle = 0, \tag{19.32}$$

or, if we allow for (19.6),

$$\omega (c_{k0} - c_{k3}) |\Phi\rangle = 0. \tag{19.33}$$

For $\omega \neq 0$ the last condition, when combined with (19.31), leads to

$$(c_{k0} - c_{k3}) |\Phi\rangle = 0, \quad \langle\Phi| (c_{k0}^{+} - c_{k3}^{+}) = 0, \tag{19.34}$$

which conditions the state vectors of the quantized electromagnetic field must satisfy. Then, as a corollary to these conditions we obtain

$$\langle\Phi| c_{k0}^{+} c_{k0} - c_{k3}^{+} c_{k3} |\Phi\rangle \equiv \frac{1}{2} \langle\Phi| (c_{k0}^{+} + c_{k3}^{+}) (c_{k0} - c_{k3}) + (c_{k0}^{+} - c_{k3}^{+}) (c_{k0} + c_{k3}) |\Phi\rangle = 0,$$

that is,

$$\langle\Phi| c_{k0}^{+} c_{k0} |\Phi\rangle = \langle\Phi| c_{k3}^{+} c_{k3} |\Phi\rangle. \tag{19.35}$$

Thanks to this property, the contributions provided by the temporal and longitudinal photons to the value of the field energy averaged over the states $|\Phi\rangle$ of the field cancel out, and we find that

$$E = \langle H \rangle = \frac{1}{2} \sum_{k} \omega \langle c_{k1}^{+} c_{k1} + c_{k1} c_{k1}^{+} \rangle + \frac{1}{2} \sum_{k} \omega \langle c_{k2}^{+} c_{k2} + c_{k2} c_{k2}^{+} \rangle. \tag{19.36}$$

This value is, obviously, positive definite for those states that contain neither longitudinal nor temporal photons. Such physical states, which possess a positive norm (see (19.27)), we will use in what follows. Let us eliminate the indefinitely large part, proportional to $\Sigma\omega$, by requiring that the creation and annihilation operators follow in normal order. As a result we arrive at the following expression for the eigenvalues of the field energy:

$$E = \sum_{k,\, s=1,\, 2} N_{ks} \omega, \tag{19.37}$$

which coincide with the one found in Section 13 for the case of Coulomb (three-dimensional transverse) gauge.

Note that conditions (19.30) and (19.31) ensure that the Lorentz condition is satisfied in the mean:

$$\left\langle \frac{\partial A^{\mu}}{\partial x^{\mu}} \right\rangle = \left\langle \left(\frac{\partial A^{\mu}}{\partial x^{\mu}} \right)^{(+)} + \left(\frac{\partial A^{\mu}}{\partial x^{\mu}} \right)^{(-)} \right\rangle = 0. \tag{19.38}$$

The same conditions, obviously, lead to the Maxwell equations in the mean. Indeed, since the field operators A_μ satisfy d'Alembert's equation, we have

$$\Box A^\mu = \frac{\partial^2 A^\mu}{\partial x^\nu \partial x_\nu} \equiv \frac{\partial}{\partial x^\nu}\left(\frac{\partial A^\mu}{\partial x_\nu} - \frac{\partial A^\nu}{\partial x_\mu}\right) + \frac{\partial}{\partial x^\nu}\frac{\partial A^\nu}{\partial x_\mu}$$
$$= \frac{\partial F^{\nu\mu}}{\partial x^\nu} + \frac{\partial}{\partial x_\mu}\frac{\partial A^\nu}{\partial x^\nu} = 0.$$

Combining this with (19.38) yields

$$\left\langle \frac{\partial F^{\nu\mu}}{\partial x^\nu} \right\rangle = 0, \tag{19.39}$$

which are the Maxwell equations for the free electromagnetic field.

Thus, longitudinal and temporal photons cannot be observed and contribute nothing to the energy and momentum of the field,

$$E = \sum_{\mathbf{k},\, s=1,\, 2} \omega N_{\mathbf{k}s} \geqslant 0, \quad \mathbf{P} = \sum_{\mathbf{k},\, s=1,\, 2} \mathbf{k} N_{\mathbf{k}s}. \tag{19.40}$$

But why cannot we reason along the same lines as we did in the classical theory (see Section 10), that is, impose the Lorentz condition directly on the field operators A_μ. The answer is as follows: such a condition would have led to a dependence between the operator coefficients $c_{\mathbf{k}0}$ and $c_{\mathbf{k}3}$, say, $c_{\mathbf{k}0} = c_{\mathbf{k}3}$, and this contradicts the commutation relations (19.17) and (19.21).

A final remark is in order. Since longitudinal and temporal photons cannot be observed, we could redefine the initial vacuum in such a manner that it would contain an arbitrary number of temporal photons and also satisfy the conditions

$$c_{\mathbf{k}1} \mid \mathrm{vac}\rangle = c_{\mathbf{k}2} \mid \mathrm{vac}\rangle = 0, \quad (c_{\mathbf{k}0} - c_{\mathbf{k}3}) \mid \mathrm{vac}\rangle = 0. \tag{19.41}$$

Such a new vacuum $\mid \mathrm{vac}\rangle$ can be obtained from the initial vacuum $\mid 0\rangle$, which contains no photons, via a gauge transformation U:

$$\mid \mathrm{vac}\rangle = U \mid 0\rangle. \tag{19.42}$$

The explicit form of operator U will not be given here, but it is important to note that

$$U = U\,(c_{\mathbf{k}0} - c_{\mathbf{k}3},\ c^+_{\mathbf{k}0} - c^+_{\mathbf{k}3})$$

and $[U,\ c_{\mathbf{k}0} - c_{\mathbf{k}3}] = 0$, so that Eqs. (19.41) are satisfied.

Let us assume that there exists a state of the field with a single photon specified by a polarization fixed by the 4-vector e_μ. Then the gauge condition (19.32) yields

$$e^\mu k_\mu = 0. \tag{19.43}$$

This means, for one, that there can be no pure states with a single temporal photon $e^\mu = (1, 0, 0, 0)$ or a single longitudinal photon $e^\mu = (0, 0, 0, 1)$. A photon whose polarization satisfies condition (19.43) is known as a four-dimensional transversal photon. This condition is not violated if the 4-vector of polarization is subjected to a gradient (or gauge) transformation

$$e_\mu \to e'_\mu = e_\mu + fk_\mu, \tag{19.44}$$

where f is an arbitrary scalar function, and $k^2 = k_\mu k^\mu = 0$. In particular, we can select f in such a way that in a given reference frame we obtain the condition of three-dimensional transversality

$$e_0 = 0, \ \mathbf{e}\cdot\mathbf{k} = 0, \ \mathbf{e}^2 = 1 \tag{19.45}$$

instead of (19.43). As the reader can easily see, the normalization condition of the unit 4-vector e^μ,

$$e_\mu e^\mu = -1 \tag{19.46}$$

does not change under (19.44).

20. The Commutation and Causal Functions of the Electromagnetic Field

Now we turn to the study of quadratic combinations of electromagnetic-field operators $A_\mu(x)$ and their averages, which are expressed in terms of various singular solutions of the wave equation for the electromagnetic potentials.

20.1 The Commutation Relation for Electromagnetic-Field Operators

Let us write the electromagnetic-field operator, following (19.12) and (19.19), in the form

$$A_\mu(x) = \frac{1}{L^{3/2}} \sum_{\mathbf{k}\lambda} \sqrt{\frac{2\pi}{\omega}}\, e_\mu^{(\lambda)} (c_{\mathbf{k}\lambda}^+ e^{ikx} + c_{\mathbf{k}\lambda}^- e^{ikx}). \tag{20.1}$$

The photon creation and annihilation operators, c^+ and c, in which the operator A_μ is expanded, satisfy the commutation relations (19.17) and (19.21). We now wish to calculate the commutator of the field operators A_μ at points x and x', or $[A_\mu(x), A_\nu(x')]$. To this end, we employ expansion (20.1) and the commutators involving c and c^+,

$$[c_{\mathbf{k}\lambda}, c^+_{\mathbf{k}'\lambda'}] = -\delta_{\mathbf{k}\mathbf{k}'} g_{\lambda\lambda'}. \tag{20.2}$$

Then

$$[A_\mu(x),\ A_\nu(x')] = \frac{2\pi}{L^3}\sum_{\mathbf{k}}\sum_{\lambda\lambda'}\frac{1}{\omega}e_\mu^{(\lambda)}e_\nu^{(\lambda')}g_{\lambda\lambda'}\left(e^{ik(x-x')} - e^{-ik(x-x')}\right).$$

Here, as it follows from (19.5), the sum over the polarizations yields

$$\sum_{\lambda\lambda'} e_\mu^{(\lambda)}e_\nu^{(\lambda')}g_{\lambda\lambda'} = g_{\mu\nu},$$

so that

$$[A_\mu(x),\ A_\nu(x')] = \frac{2\pi}{L^3}g_{\mu\nu}\sum_{\mathbf{k}}\frac{1}{\omega}\left(e^{ik(x-x')} - e^{-ik(x-x')}\right). \tag{20.3}$$

Going over from summation to integration via

$$\frac{1}{L^3}\sum_{\mathbf{k}} = \frac{1}{(2\pi)^3}\int d^3k \tag{20.4}$$

and replacing $\mathbf{k}$ with $-\mathbf{k}$ in the first integral, we find that

$$[A_\mu(x),\ A_\nu(x')] = 4\pi i g_{\mu\nu}D_0(x-x'), \tag{20.5}$$

where

$$D_0(x-x') = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\omega}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\sin\omega(t-t'). \tag{20.6}$$

The invariant function $D_0(x-x')$ is known as the commutation function for the electromagnetic field. Since $A_\mu(x)$ and $A_\nu(x')$ satisfy homogeneous d'Alembert's equation, so does $D_0(x-x')$:

$$\Box_x D_0(x-x') = \Box_{x'} D_0(x-x') = 0. \tag{20.7}$$

As definition (20.6) implies, the function $D_0(x) = D_0(\mathbf{r}, t)$, being the solution to Eq. (20.7), satisfies the following initial conditions:

$$D_0(\mathbf{r}, 0) = 0, \quad \frac{\partial}{\partial t}D_0(\mathbf{r}, t)|_{t=0} = \delta(\mathbf{r}). \tag{20.8}$$

Hence, at $t = t'$ Eq. (20.5) yields the equal-time commutator

$$\left[A_\mu(x),\ \frac{\partial}{\partial t'}A_\nu(x')\right]_{t=t'} = -4\pi i g_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}'). \tag{20.9}$$

The meaning of this relationship can easily be clarified if we go over to the canonical field-momentum operator

$$\pi_\nu = -\frac{1}{4\pi}\frac{\partial A_\nu}{\partial t}.$$

Then instead of (20.9) we get

$$[A_\mu(x),\ \pi_\nu(x')]_{t=t'} = ig_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}'). \tag{20.10}$$

The operators A_μ and π_ν are canonically conjugate quantities and correspond to the generalized coordinate and momentum of the continuous system, the electromagnetic field. Hence, the commutator (20.10) constitutes a direct generalization of the commutation relation (18.1) for a quantum mechanical discrete system. Note that we could also put (20.10) at the base of the quantum theory of the electromagnetic field, but the substantiation of this step (see Section 18) would require partitioning the space into cells and thus going over to a discrete system, which procedure is in itself rather awkward.

Let us now show that the commutation function $D_0(x)$ is relativistically invariant. To this end, we evaluate in (20.6) the integral with respect to **k**. The result is

$$D_0(x) = \frac{2\pi}{(2\pi)^3} \int_0^\infty \omega\, d\omega \int_0^\pi \sin\theta\, d\theta e^{i\omega r\cos\theta} \sin\omega t$$

$$= \frac{1}{2ir\,(2\pi)^2} \int_{-\infty}^{\infty} d\omega\,(e^{i\omega r} - e^{-i\omega r}) \sin\omega t.$$

Integration with respect to ω yields

$$D_0(x) = \frac{1}{4\pi r}[\delta(r-t) - \delta(r+t)]. \tag{20.11}$$

Since r is always nonnegative, we have

$$\delta(r^2 - t^2) = \frac{\delta(r-|t|)}{2r}.$$

The final result is

$$D_0(x) = \frac{1}{2\pi}\delta(x^2)\operatorname{sgn} x^0, \tag{20.12}$$

where $\operatorname{sgn} x^0$ is the sign function (see (13.27)).

As is known (see Section 2), the sign of $t = x^0$, that is, $\operatorname{sgn} x^0$, is invariant for the isotropic vector $x^2 = 0$, which proves the invariance of $D_0(x)$. The reader will recall that in Section 13 we introduced the singular solutions of the inhomogeneous wave equation, $D_{\text{ret}}(x)$ and $D_{\text{adv}}(x)$, which are the retarded and advanced Green functions (13.17) and (13.22), as well as their half-difference D_{rad} given in (13.26).

Comparing these expressions with (20.12), we get

$$D_0(x) = D_{\text{ret}}(x) - D_{\text{adv}}(x) = 2D_{\text{rad}}(x). \tag{20.13}$$

Let us now find the vacuum average of quadratic combinations of potentials, $A_\mu(x)\,A_\nu(x')$. Using expansion (20.1) and the property of the vacuum, we get

$$\langle 0 \mid A_\mu(x)\,A_\nu(x') \mid 0\rangle = -4\pi g_{\mu\nu} D_+(x - x'), \tag{20.14}$$

where

$$D_+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{-ik(x-x')}, \tag{20.15}$$

and, similarly,

$$\langle 0 \mid A_\nu(x') A_\mu(x) \mid 0 \rangle = -4\pi g_{\mu\nu} D_-(x - x'), \tag{20.16}$$

where

$$D_-(x) = D_+(-x) = D_+^*(x). \tag{20.17}$$

Averaging (20.5) over the vacuum, we arrive at a relationship that links the three functions:

$$D_0(x) = i(D_+(x) - D_-(x)). \tag{20.18}$$

The functions D_+, D_-, and D_0 are singular solutions of the homogeneous wave equation. It can be shown that these invariant functions have the following form:

$$D_\pm(x) = -\frac{1}{(2\pi)^2 x^2} \mp \frac{i}{4\pi} \delta(x^2) \operatorname{sgn} x^0. \tag{20.19}$$

20.2 The Normal and Chronologically Ordered Products of Electromagnetic-Field Operators

In Section 18 we introduced the concept of normal product of photon creation and annihilation operators. The definition we gave is also valid for the field operators $A_\mu(x)$. Precisely, we define the normal product of two operators, $A_\mu(x)$ and $A_\nu(x')$, in such a manner that all creation operators are to the left of all the annihilation operators. Since we know that $A_\mu(x)$ can be decomposed in an invariant manner,

$$A_\mu(x) = A_\mu^{(-)}(x) + A_\mu^{(+)}(x), \tag{20.20}$$

where, as (20.1) implies, the negative-frequency part $A_\mu^{(-)}(x)$ contains only creation operators and the positive-frequency part $A_\mu^{(+)}$, only annihilation operators, the normal product (denoted by the letter N) is

$$\begin{aligned} NA_\mu(x) A_\nu(x') = {} & A_\mu^{(+)}(x) A_\nu^{(+)}(x') + A_\mu^{(-)}(x) A_\nu^{(-)}(x') \\ & + A_\mu^{(-)}(x) A_\nu^{(+)}(x') + A_\nu^{(-)}(x') A_\mu^{(+)}(x). \end{aligned} \tag{20.21}$$

We now introduce another ordering scheme for field operators, namely, the chronologically ordered product. We define the chronologically ordered product, or T-product, as follows:

$$TA_\mu(x) A_\nu(x') = \begin{cases} A_\mu(x) A_\nu(x'), & t > t', \\ A_\nu(x') A_\mu(x), & t' > t, \end{cases} \tag{20.22}$$

where the operators are placed in the order of increasing times from right to left.

Just as in the case of the normal product, the chronologically ordered product is defined in an invariant manner. Indeed, for $(x - x')^2 \geqslant 0$ the sign of $t - t'$ is invariant, while for $(x - x')^2 < 0$ the operators $A(x)$ and $A(x')$ are commutative.

The definitions of the T- and the N-product of any number of operators can be given in a similar manner. Note that the reason why the N-product was introduced is that the average of the N-product of any number of operators over the vacuum is zero:

$$\langle 0 \mid NA_\mu(x)\, A_\nu(x')\, A_\lambda(x'') \ldots \mid 0\rangle = 0. \tag{20.23}$$

Let us define the contraction, or chronological pairing, of two operators as follows:

$$\overbrace{A_\mu(x)\, A_\nu(x')} = TA_\mu(x)\, A_\nu(x') - NA_\mu(x)\, A_\nu(x'). \tag{20.24}$$

We can easily see that this is a c-number, that is, not an operator. Suppose that $x^0 > x'^0$. Then

$$\begin{aligned} TA_\mu(x)\, A_\nu(x') - NA_\mu(x)\, A_\nu(x') &= A_\mu^{(+)}(x)\, A_\nu^{(-)}(x') - A_\nu^{(-)}(x')\, A_\mu^{(+)}(x) \\ &= -4\pi g_{\mu\nu} D_+(x - x'), \end{aligned} \tag{20.25}$$

and similarly for $x^0 < x'^0$. Averaging (20.24) over the vacuum and employing (20.23), we find that the pairing is the average of the T-product over the vacuum state:

$$\overbrace{A_\mu(x)\, A_\nu(x')} = \langle 0 | TA_\mu(x)\, A_\nu(x') | 0\rangle \equiv \langle TA_\mu(x)\, A_\nu(x')\rangle_0, \tag{20.26}$$

whence, employing (20.14) and (20.16), we get

$$\overbrace{A_\mu(x)\, A_\nu(x')} = \begin{cases} \langle A_\mu(x)\, A_\nu(x')\rangle_0 = -4\pi g_{\mu\nu} D_+(x - x'), & t > t', \\ \langle A_\nu(x')\, A_\mu(x)\rangle_0 = -4\pi g_{\mu\nu} D_-(x - x'), & t' > t. \end{cases} \tag{20.27}$$

We introduce the function

$$D_c(x) = \theta(x)\, D_+(x) + \theta(-x)\, D_-(x), \tag{20.28}$$

which is known as the causal function, or propagator. The pairing is then expressed in terms of the propagator as follows:

$$\overbrace{A_\mu(x)\, A_\nu(x')} = \langle TA_\mu(x)\, A_\nu(x')\rangle_0 = -4\pi g_{\mu\nu} D_c(x - x'). \tag{20.29}$$

From (20.27) and (20.28) it follows that a propagator describes the following chain of events: for $t > t'$ a photon is created at point x' and is then destroyed at point x, while for $t < t'$ the order is reversed, that is, a photon is created at point x and destroyed at point x'.

The integral representation for $D_c(x)$ follows from the definitions (20.15) and (20.17) for $D_+(x)$ and $D_-(x)$, respectively, where in the integral for $D_-(x)$ we must replace $\mathbf{k}$ with $-\mathbf{k}$:

$$D_c(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega} e^{i\mathbf{k}\cdot\mathbf{r} - i\omega|t|}. \tag{20.30}$$

The propagator can be written in the form of a quadruple integral,

$$D_c(x) = \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 + i\varepsilon} e^{-ikx}, \tag{20.31}$$

where $\varepsilon \to +0$, that is, is a particular solution of the inhomogeneous wave equation

$$\Box D_c(x) = -i\delta(x). \tag{20.32}$$

Some particular solutions of such an equation (without the factor $-i$ in front of the delta function) were discussed in Section 13. As was established there the particular solutions differ in the choice of

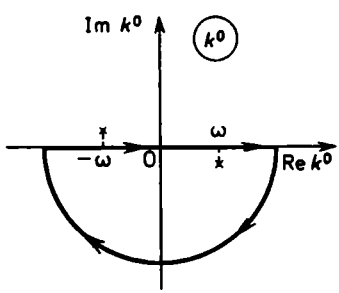

Figure 20.1

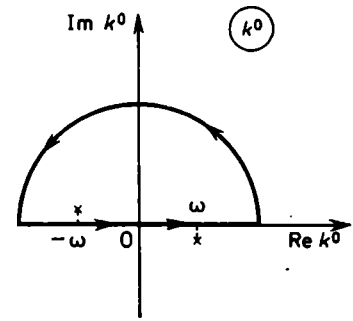

Figure 20.2

the sense of traversal of the poles of the integrand of the integral representation. Introduction of a small imaginary term $i\varepsilon$ in (20.31) means that the integral with respect to k^0 contains two poles in the integrand at $k^0 = \pm(\omega - i\varepsilon)$, $\omega = |\mathbf{k}|$, one of which (the left one) lies above the real axis, and the other (the right one) lies below the real axis. If $t > 0$, the integration contour is closed in the lower half-plane (Figure 20.1), while if $t < 0$, the integration contour is closed in the upper half-plane (Figure 20.2). As a result we find that

$$D_c(x) = D_+(x), \; t > 0, \; D_c(x) = D_-(x), \; t < 0. \tag{20.33}$$

We have thus proved that (20.31) indeed coincides with definition (20.28). Note that, as follows from (20.13) and the definitions (13.20) and (13.25) for D_{ret} and D_{adv} in terms of contour integrals (see Figure 13.1), the commutation function $D_0(x)$ can also be written as a contour integral:

$$D_0(x) = -\frac{1}{(2\pi)^4} \int_{C_0} \frac{d^4k}{k^2} e^{-ikx}, \quad (20.34)$$

where contour C_0 is depicted in Figure 20.3.

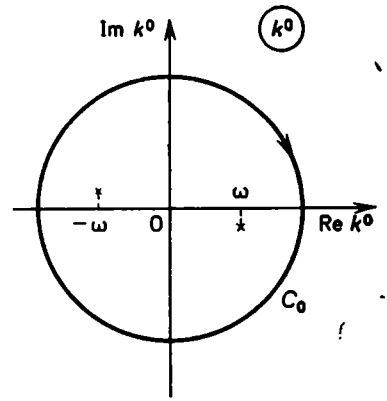

Figure 20.3

A final important remark is in order. The expression for the commutation function (20.5) was obtained through expanding the electromagnetic-potential operators, (20.1), in a gauge specified by (19.5). In accordance with (19.44), we could have also considered other gauges for which

$$e_\mu^{(\lambda)} \to e_\mu'^{(\lambda)} = e_\mu^{(\lambda)} + f^{(\lambda)} k_\mu, \quad (20.35)$$

where $f^{(\lambda)}$ are arbitrary scalar functions, with $k^2 \neq 0$, generally speaking. In this case the commutation function (20.5) is replaced with functions of the following type:

$$D_{\mu\nu}(x) = g_{\mu\nu} D_0(x) \to D'_{\mu\nu}(x) = g_{\mu\nu} D_0(x) + \frac{1}{(2\pi)^4} \int_{C_0} \frac{d^4k}{k^2} e^{-ikx} (f_\nu k_\mu + f_\mu k_\nu + f k_\mu k_\nu), \quad (20.36)$$

where

$$f_\nu = \sum_{\lambda\lambda'} g_{\lambda\lambda'} f^{(\lambda)} e_\nu^{(\lambda')}, \quad f_\mu = \sum_{\lambda\lambda'} g_{\lambda\lambda'} f^{(\lambda')} e_\mu^{(\lambda)}, \quad f = \sum_{\lambda\lambda'} g_{\lambda\lambda'} f^{(\lambda)} f^{(\lambda')}. \quad (20.37)$$

In the same manner, the gauge transformation results in a change in propagator:

$$D^c_{\mu\nu}(x) = g_{\mu\nu} D_c(x) \to g_{\mu\nu} D_c(x) + \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 + i\varepsilon} e^{-ikx} (f_\nu k_\mu + f_\mu k_\nu + f k_\mu k_\nu). \quad (20.38)$$

Say, if we put $f^{(\lambda)} = -k^\lambda / k^2$, we arrive at the following formula for the propagator:

$$D^c_{\mu\nu}(x) = \frac{i}{(2\pi)^4} \int \frac{d^4k}{k^2 + i\varepsilon} e^{-ikx} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \quad (20.39)$$

The concrete choice of gauge is determined by the features of the problem being solved. At the same time, in view of gauge invariance, the final results do not depend on this choice.

21. Quantization of the Dirac Field

The Dirac equation describes massive charged particles with half-integral spin, electrons and positrons. Quantization of the electron-positron field differs from quantization of the electromagnetic field primarily in that particles described by the Dirac equation are fermions, that is, obey the Fermi-Dirac statistics. This fact is also expressed by Pauli's exclusion principle: there can be no more than one fermi-particle in a single quantum state.

21.1 Charge Conjugation

The free electron is described by the Dirac equation (8.40), which after being multiplied by the matrix γ^0 can be expressed in the following Hamilton form:

$$i\frac{\partial}{\partial t}\psi(\mathbf{r}, t) = \mathrm{H_D}\psi(\mathbf{r}, t), \tag{21.1}$$

where

$$\mathrm{H_D} = -i\boldsymbol{\alpha}\cdot\nabla + \beta m \tag{21.2}$$

is the Hamiltonian of the Dirac equation, and $\boldsymbol{\alpha}$ and $\beta = \gamma^0$ are the Dirac matrices (see Section 8). We represent the Dirac wave function of a stationary state by

$$\psi(\mathbf{r}, t) = Ae^{-iEt+i\mathbf{r}\cdot\mathbf{p}}u, \tag{21.3}$$

where E is an eigenvalue of $\mathrm{H_D}$, $\mathbf{p}$ is an eigenvalue of the momentum operator $-i\nabla = \mathbf{p}$, u is a constant bispinor, and A is the normalization constant.

Let us write the bispinor u in terms of the two-component spinors φ and χ,

$$u = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{21.4}$$

and use the standard representation for the Dirac matrices (8.48) and (8.52). Then for φ and χ we have the system of equations

$$\begin{aligned} (E - m)\,\varphi &= (\boldsymbol{\sigma}\cdot\mathbf{p})\,\chi, \\ (E + m)\,\chi &= (\boldsymbol{\sigma}\cdot\mathbf{p})\,\varphi, \end{aligned} \tag{21.5}$$

whose compatibility is ensured by the condition

$$E^2 = \mathbf{p}^2 + m^2.$$

This leads to a relationship that links the eigenvalues of the Hamiltonian operator and the momentum:

$$E = \pm\varepsilon_{\mathbf{p}}, \quad \varepsilon_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}. \tag{21.6}$$

Thus, the possible solutions split into two branches, the positive-frequency branch with $E = \varepsilon_{\mathbf{p}}$ and the negative-frequency branch with $E = -\varepsilon_{\mathbf{p}}$, and

$$\psi^{(+)} \propto e^{-i\varepsilon_{\mathbf{p}} t}, \quad \psi^{(-)} \propto e^{i\varepsilon_{\mathbf{p}} t}. \tag{21.7}$$

Since the two branches correspond to different eigenvalues of the Hamiltonian, the corresponding wave functions must be orthogonal to each other:

$$\int (\psi^{(+)})^+ \psi^{(-)} d^3x = 0. \tag{21.8}$$

Note that the separation (21.7) is Lorentz-invariant, since the 4-vector $p^\mu = (E, \mathbf{p})$ is timelike: $p^2 = E^2 - \mathbf{p}^2 = m^2 > 0$.

If the spinor φ is given, the other spinor χ can be determined from the second equation of the system (21.5):

$$\chi = \frac{(\sigma \cdot \mathbf{p})}{E + m} \varphi. \tag{21.9}$$

Specifying spinor is accomplished by imposing an additional condition that fixes the spin state of the particle. For instance, we can require that the wave function be an eigenfunction of the helicity operator* $(1/2)(\Sigma \cdot \mathbf{p})$:

$$\frac{1}{2}(\Sigma \cdot \mathbf{p})\psi = \sigma |\mathbf{p}| \psi, \quad \sigma = \pm\frac{1}{2}. \tag{21.10}$$

This operator, obviously, is commutative with H_D, and therefore the Dirac equation and Eq. (21.10) have common solutions.

The solutions to the Dirac equation for a free particle, (21.1), corresponding to two different branches, $E = \varepsilon_{\mathbf{p}}$ and $E = -\varepsilon_{\mathbf{p}}$, can be written finally as follows:

$$\psi_{\mathbf{p}}^{(+)}(\mathbf{r}, t) = \frac{1}{\sqrt{2\varepsilon_{\mathbf{p}} V}} u(\sigma, p) e^{-ipx}, \tag{21.12}$$

$$\psi_{\mathbf{p}}^{(-)}(\mathbf{r}, t) = \frac{1}{\sqrt{2\varepsilon_{\mathbf{p}} V}} u(\sigma, -p) e^{ipx}. \tag{21.13}$$

---

* This operator is proportional to the temporal component of the 4-vector spin operator

$$S^\mu = (S^0, \mathbf{S}), \quad S^0 = \frac{1}{m}\Sigma \cdot \mathbf{p}, \quad \mathbf{S} = \frac{1}{m}(m\gamma^0\Sigma - \mathbf{p}\gamma^5), \tag{21.11}$$

where $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv -\rho_1$ is the pseudoscalar Dirac matrix. A detailed description of spin operators is given in [2].

Here V is the normalization volume, and $p^\mu = (p^0, \mathbf{p})$ is the 4-momentum, where $p^0 \equiv \varepsilon_\mathbf{p}$, with the negative-frequency solution $\psi^{(-)}$ chosen in such a way that it corresponds to the value of momentum $-\mathbf{p}$ taken with a sign opposite to that of $\psi^{(+)}$. The bispinors $u(\sigma, \pm p)$ satisfy the equations

$$(\pm \gamma p - m)\, u(\sigma, \pm p) = 0. \tag{21.14}$$

The normalization coefficient $A = (2\varepsilon_\mathbf{p} V)^{-1/2}$ is selected such that the bispinors are normalized by the conditions

$$\bar{u}(p)\, \gamma^\mu u(p) = 2p^\mu, \quad \bar{u}(-p)\, \gamma^\mu\, u(-p) = 2p^\mu. \tag{21.15}$$

Then at $\mu = 0$ we get

$$u^+(p)\, u(p) = 2\varepsilon_\mathbf{p}, \quad u^+(-p)\, u(-p) = 2\varepsilon_\mathbf{p}, \tag{21.16}$$

whence

$$\int_V (\psi_\mathbf{p}^{(+)})^+ \psi_\mathbf{p}^{(+)} d^3x = 1, \qquad \int_V (\psi_\mathbf{p}^{(-)})^+ \psi_\mathbf{p}^{(-)} d^3x = 1.$$

The positive-frequency solutions of the Dirac equation, (21.12), can be interpreted as being the wave functions of an electron whose energy is $E = \varepsilon_\mathbf{p} > 0$. To give meaning to the negative-frequency branch of solutions, (21.13), with $E = -\varepsilon_\mathbf{p}$, we consider the so-called charge conjugation operation, or C-conjugation, to which the Dirac functions can be subjected. Let us assume that the Dirac particles are not free but interact with the electromagnetic field $A_\mu(x)$. In this case the Dirac equation is obtained by "extending" the derivative $\partial_\mu \to \partial_\mu + ieA_\mu$ in the same manner as was done in Section 10:

$$(i\gamma\partial - e\gamma A - m)\, \psi(x) = 0, \qquad \bar{\psi}(x)\, (i\gamma\partial + e\gamma A + m) = 0. \tag{21.17}$$

We introduce the charge conjugation operation as follows

$$\begin{aligned} \psi(x) &\to \psi^C(x) = C\bar{\psi}(x), \\ \bar{\psi}(x) &\to \bar{\psi}^C(x) = C^{-1}\psi(x), \\ A_\mu(x) &\to A_\mu^C(x) = -A_\mu(x). \end{aligned} \tag{21.18}$$

Suppose that the operator C possesses the following property:

$$C^{-1}\gamma^\mu C = -\gamma_\mu^T. \tag{21.19}$$

Then, obviously, Eqs. (21.17) will remain invariant under C-conjugation (21.18):

$$(i\gamma\partial - e\gamma A^C - m)\, \psi^C(x) = 0, \quad \bar{\psi}^C(x)\, (i\gamma\partial + e\gamma A^C + m) = 0. \tag{21.20}$$

Condition (21.19) can be met if we select the C-conjugation operator in the form $C = \gamma^2\gamma^0$ (refer to the properties of Dirac matrices discussed in Section 8). Matrix C possesses the following properties:

$$C^+ = C^{-1}, \quad C^T = -C, \tag{21.21}$$

that is, it is unitary and skew-symmetric.

Using the charge conjugation operation, we can construct a positive-frequency part out of the negative-frequency part of the solution:

$$\psi^{C(+)}(x) = C\overline{\psi^{(-)}(x)}. \tag{21.22}$$

As (21.20) implies, the function $\psi^{C(+)}$ satisfies the same equation as $\psi^{(+)}$, but with the opposite sign of the charge (since $A^C = -A$) and has a positive frequency, that is, the normal sign of the energy $E = \varepsilon_p > 0$. This function, $\psi^{C(+)}$, describes a new particle, the positron, that is, the antiparticle of the electron, which carries a charge of opposite sign. If the electron charge is negative, $e = -e_0 < 0$, the charge of the positron is positive, $e = e_0 > 0$. Thus, the wave function of the electron, ψ_{el}, and that of the positron, ψ_{pos}, are constructed from the positive- and negative-frequency parts of the solution to the Dirac equation,

$$\psi_{el} = \psi^{(+)}, \quad \psi_{pos} = C\overline{\psi^{(-)}}. \tag{21.23}$$

21.2 Commutation Relations. Anticommutators

Now let us consider the Dirac equation (9.1) not as a quantum mechanical equation for single-particle states but as an equation for a spinor Dirac field. The general solution for the free case, $A_\mu(x) = 0$, can be split in an invariant manner into the positive- and negative-frequency parts $\psi^{(+)}$ and $\psi^{(-)}$ as follows:

$$\psi(x) = \psi^{(+)}(x) + \psi^{(-)}(x) = \sum_s (b_s\psi_s^{(+)}(x) + b_s^*\psi_s^{(-)}(x)), \tag{21.24}$$

where s stands for the set of quantum numbers specifying the stationary states

$$\psi_s^{(+)} \propto e^{-i\varepsilon_s^{(+)}t}, \quad \psi_s^{(-)} \propto e^{i\varepsilon_s^{(-)}t}, \tag{21.25}$$

$\pm\varepsilon_s^{(\pm)}$ are the eigenvalues of the Dirac Hamiltonian, and a_s and b_s^* are expansion coefficients.

Let us find the total energy and charge of the field (see Section 9):

$$H=\int d^3x\,\mathcal{H},\quad \mathcal{H}=\frac{i}{2}(\bar{\psi}\gamma^0\psi_{,t}-\bar{\psi}_{,t}\gamma^0\psi)=\frac{i}{2}(\psi^+\psi_{,t}-\psi^+_{,t}\psi),$$
$$Q=e\int d^3x\bar{\psi}\gamma^0\psi=e\int d^3x\psi^+\psi. \tag{21.26}$$

Substituting expansion (21.24) and employing the fact that the solutions $\psi_s^{(\pm)}$ are orthonormalized, we obtain

$$H=\sum_s(\varepsilon_s^{(+)}a_s^*a_s-\varepsilon_s^{(-)}b_sb_s^*),\quad Q=e\sum_s(a_s^*a_s+b_sb_s^*). \tag{21.27}$$

As we see, the energy of the field does not possess a definite sign, while the charge has the sign of e. The reader will recall that the expansion coefficients $a_{\mathbf{k}s}$ were subjected, when the electromagnetic field was quantized, to commutation relations (18.16) so that they would become operators. We will now define the operators of energy and charge for the Dirac field using the classical expressions (21.27) as a model, replacing a_s and b_s in these expressions with operators. The result is

$$\mathrm{H}=\sum_s(\varepsilon_s^{(+)}\mathrm{a}_s^+\mathrm{a}_s-\varepsilon_s^{(-)}\mathrm{b}_s\mathrm{b}_s^+),\quad \mathrm{Q}=e\sum_s(\mathrm{a}_s^+\mathrm{a}_s+\mathrm{b}_s\mathrm{b}_s^+), \tag{21.28}$$

where a_s^+ and b_s^+ are the hermitian-conjugate operators. However, in contrast to the electromagnetic field (a boson field!), we cannot require the validity of the commutation relations of the (18.16) type for a_s and b_s, since if these relations were valid, the energy would not be positive definite and the electrons and positrons would contribute terms of the same sign to the charge of the field. Consequently, for the quantized expansion coefficients of the operator of the Dirac field

$$\psi(x)=\sum_s(\mathrm{a}_s\psi_s^{(+)}(x)+\mathrm{b}_s^+\psi_s^{(-)}(x)),$$
$$\bar{\psi}(x)=\sum_s(\mathrm{a}_s^+\overline{\psi_s^{(+)}}(x)+\mathrm{b}_s\overline{\psi_s^{(-)}}(x)), \tag{21.29}$$

that is, operators a_s and b_s, we postulate the following commutation relations given in the form of anticommutators:

$$\{\mathrm{a}_s,\ \mathrm{a}_{s'}^+\}\equiv\mathrm{a}_s\mathrm{a}_{s'}^++\mathrm{a}_{s'}^+\mathrm{a}_s=\delta_{ss'},\quad \{\mathrm{a}_s,\ \mathrm{a}_{s'}\}=\{\mathrm{a}_s^+,\ \mathrm{a}_{s'}^+\}=0, \tag{21.30}$$

and similarly for operators b and b^+:

$$\{\mathrm{b}_s,\ \mathrm{b}_{s'}^+\}=\delta_{ss'},\ \text{etc.}, \tag{21.31}$$

with

$$\{\mathrm{a}_s,\ \mathrm{b}_{s'}\}=\{\mathrm{a}_s^+,\ \mathrm{b}_{s'}\}=\{\mathrm{a}_s,\ \mathrm{b}_{s'}^+\}=0. \tag{21.32}$$

Now let us define the operators

$$N_s^{(+)} = a_s^+ a_s, \quad N_s^{(-)} = b_s^+ b_s. \tag{21.33}$$

By virtue of the commutation relations (21.30),

$$(N_s^{(+)})^2 = a_s^+ a_s a_s^+ a_s = -(a_s^+)^2 (a_s)^2 + a_s^+ a_s = N_s^{(+)}, \tag{21.34}$$

that is, the operator $N_s^{(+)}$ has only two eigenvalues, $N_s^{(+)} = 0, 1$, and similarly, $N_s^{(-)} = 0, 1$.

When describing the states of the Dirac field in the occupation number space, that is, using the language of particles, it is natural to assume that $N_s^{(+)}$ and $N_s^{(-)}$ are the electron and positron occupation number operators, under the assumption that $\psi_s^{(+)}$ and $\psi_s^{(-)}$ describe the electron and the positron, respectively. Then the conditions $N_s^{(\pm)} = 0, 1$ express the well-known Pauli exclusion principle, which limits the occupation numbers for fermion states to values not exceeding unity. If we use the commutation relations (21.30) and (21.31), we can reduce the energy and charge operators (21.28) to

$$\begin{aligned} H &= \sum_s (\varepsilon_s^{(+)} N_s^{(+)} + \varepsilon_s^{(-)} N_s^{(-)}) - \sum_s \varepsilon_s^{(-)}, \\ Q &= e \sum_s (N_s^{(+)} - N_s^{(-)}) - e \sum_{s,(-)} 1, \end{aligned} \tag{21.35}$$

where (—) under the summation sign means that we must sum only over the negative-frequency states. If we drop the c-number terms in (21.35), we arrive at the following formulas for the energy and charge values of the electron-positron field:

$$E = \sum_s (\varepsilon_s^{(+)} N_s^{(+)} + \varepsilon_s^{(-)} N_s^{(-)}), \quad Q = e \sum_s (N_s^{(+)} - N_s^{(-)}), \tag{21.36}$$

which have a definite physical meaning, with the energy positive definite and the charge consisting of electron and positron terms with opposite signs. The exclusion of c-number divergent terms is achieved by an appropriate redefinition of the operators H and Q (see Section 22 below).

Using the commutation relations for a_s and b_s, we can obtain the following matrix elements between states with a single electron $|1_s^{(+)}\rangle$ or positron $|1_s^{(-)}\rangle$ and states in which there is no electron $|0_s^{(+)}\rangle$ or positron $|0_s^{(-)}\rangle$:

$$\begin{aligned} \langle 1_s^{(+)} | a_s^+ | 0_s^{(+)} \rangle &= 1, \quad \langle 0_s^{(+)} | a_s | 1_s^{(+)} \rangle = 1, \\ \langle 0_s^{(+)} | a_s^+ | 1_s^{(+)} \rangle &= 0, \quad \langle 1_s^{(+)} | a_s | 0_s^{(+)} \rangle = 0, \end{aligned} \tag{21.37}$$

and similarly for b_s and b_s^+. Indeed, for vacuum states $|0\rangle$ we have $a|0\rangle = 0$, whereby $\langle 1 \mid a \mid 0\rangle = \langle 0 \mid a^+ \mid 1\rangle = 0$ (for the sake of brevity the subscripts are dropped). Moreover, $a_s^+ a_s^+ = 0$, whereby

$$0 = \langle 1 \mid a^+ a^+ \mid 1\rangle = \langle 1 \mid a^+ \mid 0\rangle \langle 0 \mid a^+ \mid 1\rangle + \langle 1 \mid a^+ \mid 1\rangle \langle 1 \mid a^+ \mid 1\rangle = (\langle 1 \mid a^+ \mid 1\rangle)^2,$$

whence $\langle 1 \mid a^+ \mid 1\rangle = \langle 1 \mid a \mid 1\rangle = 0$, and similarly

$$\langle 0 \mid a^+ \mid 0\rangle = \langle 0 \mid a \mid 0\rangle = 0.$$

Finally,

$$\langle 1 \mid a^+ a \mid 1\rangle = \langle 1 \mid a^+ \mid 0\rangle \langle 0 \mid a \mid 1\rangle + \langle 1 \mid a^+ \mid 1\rangle \langle 1 \mid a \mid 1\rangle = |\langle 0 \mid a \mid 1\rangle|^2 = 1,$$

whence $\langle 0 \mid a \mid 1\rangle = 1$, etc.

Thus, a_s^+ is the electron creation operator and b_s^+ is the positron creation operator at state s, while a_s and b_s are the electron and positron annihilation operators:

$$a_s^+|0\rangle = |1_s^{(+)}\rangle, \quad b_s^+|0\rangle = |1_s^{(-)}\rangle. \tag{21.38}$$

In general, the many-particle state with $N^{(+)}$ electrons and $N^{(-)}$ positrons is obtained through acting $N^{(+)}$ times with operator a_s^+ and $N^{(-)}$ times with operator b_s^+ on the vacuum state. Note, however, that in contrast to multiphoton states, multifermion states are constructed with due regard for the order in which the operators a, b, a^+, and b^+ act (because a and b are anticommutative), which means changing the sign of the state every time the operator cofactors are permuted. For instance, if

$$|1_{s_1}^{(+)}, 0_{s_2}^{(-)}\rangle = a_{s_1}^+|0\rangle,$$

then

$$b_{s_2}^+|1_{s_1}^{(+)}, 0_{s_2}^{(-)}\rangle = -a_{s_1}^+ b_{s_2}^+|0\rangle = -|1_{s_1}^{(+)}, 1_{s_2}^{(-)}\rangle.$$

The field operators $\bar{\psi}(x)$ (21.29) unite the electron creation operator and the positron annihilation operator:

$$\langle 1_s^{(+)}|\bar{\psi}(x)|0_s^{(+)}\rangle = \overline{\psi_s^{(+)}}(x), \quad \langle 0_s^{(-)}|\bar{\psi}(x)|1_s^{(-)}\rangle = \overline{\psi_s^{(-)}}(x). \tag{21.39}$$

On the other hand, the field operators $\psi(x)$ unite the positron creation operator and the electron annihilation operator:

$$\langle 0_s^{(+)}|\psi(x)|1_s^{(+)}\rangle = \psi_s^{(+)}(x), \quad \langle 1_s^{(-)}|\psi(x)|0_s^{(-)}\rangle = \psi_s^{(-)}(x). \tag{21.40}$$

Finally, let us write out explicitly the expansion of the field Dirac operators in the operators of creation and annihilation of electrons and positrons in free states:

$$\psi(x) = \sum_{\mathbf{p},\sigma} \frac{1}{\sqrt{2\varepsilon_p V}} (a_{\mathbf{p}\sigma} u(p, \sigma) e^{-ipx} + b^+_{\mathbf{p}\sigma} u(-p, \sigma) e^{ipx}),$$
$$\bar{\psi}(x) = \sum_{\mathbf{p},\sigma} \frac{1}{\sqrt{2\varepsilon_p V}} (a^+_{\mathbf{p}\sigma} \bar{u}(p, \sigma) e^{ipx} + b_{\mathbf{p}\sigma} \bar{u}(-p, \sigma) e^{-ipx}). \quad (21.41)$$

But why for the charged Dirac field must we introduce two pairs of operators, a, a^+ and b, b^+, while the electromagnetic field requires only one pair, say, c^+ and c? The reason for this is that the bispinors $u(p)$ and $u(-p)$ satisfy different equations, while the Fourier components of the electromagnetic field $A_k^{(+)}$ and $A_k^{(-)}$ satisfy one equation. In other words, two different particles correspond to the Dirac field, the electron and the positron, while only one particle corresponds to the electromagnetic field, the photon. We note, finally, that quantization of the Dirac field via anticommutators and of the electromagnetic field via commutators ensures the positive definiteness of the field energy in both cases. At the same time, the difference in quantization of the electron-positron (fermion) field and the electromagnetic (boson) field leads to different types of statistics: in the first case this is the Fermi-Dirac statistics and its basis, Pauli's exclusion principle, while in the second case this is the Bose-Einstein statistics, with arbitrary occupation numbers. Pauli's theorem, which relates the spin (integral spin for bosons and half-integral for fermions) and statistics, manifests itself through the above-noted facts.

22. The Commutation and Causal Functions of the Dirac Field

The goal of this section is, first, to show that canonical quantization leads to the same rules of quantization of the Dirac field as the ones arrived at in the previous section and, second, to introduce the normal and chronologically ordered products. We then discuss the vacuum averages of quadratic combinations of the Dirac field operators.

22.1 Canonical Quantization

Let us write the Dirac field operators as

$$\psi(\mathbf{r}, t) = \sum_s (a_s(t) \psi_s^{(+)}(\mathbf{r}) + b_s^+(t) \psi_s^{(-)}(\mathbf{r})), \quad (22.1)$$

where $a_s(t)$ and $b_s^+(t)$ are the Heisenberg (time dependent) operators representing the coefficients in the expansion in positive-frequency and negative-frequency Dirac wave functions. For the energy operator we take definition (21.28):

$$H = \sum_s (\varepsilon_s^{(+)} a_s^+ a_s - \varepsilon_s^{(-)} b_s b_s^+). \tag{22.2}$$

Let us write the equation of motion for the Heisenberg operators $a_s(t)$ and $b_s(t)$ in the form (19.11):

$$\frac{da_s(t)}{dt} = i[H, a_s(t)], \quad \frac{db_s(t)}{dt} = i[H, b_s(t)]. \tag{22.3}$$

We will consider the first equation in greater detail:

$$\frac{da_s(t)}{dt} = \sum_{s'} (i\varepsilon_{s'}^{(+)} [a_{s'}^+ a_{s'}, a_s] - i\varepsilon_{s'}^{(-)} [b_{s'} b_{s'}^+, a_s]). \tag{22.4}$$

To transform the commutators on the right-hand side of this equation we employ the following identity, which holds true for arbitrary operators A, B and C:

$$[AB, C] = A\{B, C\} - \{A, C\} B. \tag{22.5}$$

As a result we obtain

$$\frac{da_s(t)}{dt} = i \sum_{s'} \varepsilon_{s'}^{(+)} (a_{s'}^+ \{a_{s'}, a_s\} - \{a_{s'}^+, a_s\} a_{s'})$$
$$- i \sum_{s'} \varepsilon_{s'}^{(-)} (b_{s'} \{b_{s'}^+, a_s\} - \{b_{s'}, a_s\} b_{s'}^+), \tag{22.6}$$

where the anticommutators are designated by braces.

The last operator equation can be linearized if we subject a and b to commutation relations (21.30) and (21.32). The summation signs are eliminated because of the Kronecker deltas $\delta_{ss'}$, and we find that

$$\frac{da_s}{dt} = -i\varepsilon_s^{(+)} a_s. \tag{22.7}$$

This equation has positive-frequency solutions,

$$a_s(t) = a_s(0) e^{-i\varepsilon_s^{(+)} t}. \tag{22.8}$$

In the same manner, for b_s^+ we can write the equation

$$\frac{db_s^+}{dt} = i \sum_{s'} \varepsilon_{s'}^{(+)} (a_{s'}^+ \{a_{s'}, b_s^+\} - \{a_{s'}^+, b_s^+\} a_{s'})$$
$$- i \sum_{s'} \varepsilon_{s'}^{(-)} (b_{s'} \{b_{s'}^+, b_s^+\} - \{b_{s'}, b_s^+\} b_{s'}^+), \tag{22.9}$$

which is linearized via the quantization conditions (21.31) and (21.32):

$$\frac{db_s^+}{dt} = i\varepsilon_s^{(-)} b_s^+. \tag{22.10}$$

As expected, the solutions to this equation are negative-frequency:

$$b_s^+(t) = b_s^+(0)\, e^{i\varepsilon_s^{(-)} t}. \tag{22.11}$$

Substituting solutions (22.8) and (22.11) into (22.1), we find the earlier expansions, (21.29) and (21.41).

Thus, the quantization rules fixed by the anticommutation relations (21.30)-(21.32) do, indeed, ensure that the field operators are decomposed into positive-frequency and negative-frequency parts, (22.1), (22.8), and (22.11). These rules, as the previous section has shown, also ensure that the Dirac field energy is positive definite. If instead of (22.5) we were to use (19.14a) and fix the quantization rules via commutators, then for b_s^+ we would obtain a positive-frequency solution instead of a negative-frequency one. Another consequence of this would be the indefiniteness of the sign of the field energy.

22.2 The Commutation Function

Let us consider the commutation relations for the field operators $\psi(x)$ and $\bar{\psi}(x)$. Bearing in mind the expansions (21.29) and the commutation relations (21.30)-(21.32) for a and b, we find that

$$\{\psi(x),\ \psi(x')\} = 0, \quad \{\bar{\psi}(x),\ \bar{\psi}(x')\} = 0. \tag{22.12}$$

At the same time,

$$\begin{aligned}\{\psi(x),\ \bar{\psi}(x')\} &= \sum_s \psi_s^{(+)}(x)\,\overline{\psi_s^{(+)}}(x') + \sum_s \psi_s^{(-)}(x)\,\overline{\psi_s^{(-)}}(x') \\ &= S^{(+)}(x,\ x') + S^{(-)}(x,\ x').\end{aligned} \tag{22.13}$$

For free electrons, using expansion (21.41), we get

$$\begin{aligned}S^{(+)}(x-x') &= \sum_{\mathbf{p},\sigma} \psi_{\mathbf{p}\sigma}^{(+)}(x)\,\overline{\psi_{\mathbf{p}\sigma}^{(+)}}(x') \\ &= \sum_{\mathbf{p},\sigma} \frac{1}{2\varepsilon_{\mathbf{p}} V}\, u(\sigma,\ p)\,\bar{u}(\sigma,\ p)\, e^{-ip(x-x')},\end{aligned} \tag{22.14}$$

$$\begin{aligned}S^{(-)}(x-x') &= \sum_{\mathbf{p},\sigma} \psi_{\mathbf{p},\sigma}^{(-)}(x)\,\overline{\psi_{\mathbf{p},\sigma}^{(-)}}(x') \\ &= \sum_{\mathbf{p},\sigma} \frac{1}{2\varepsilon_{\mathbf{p}} V}\, u(\sigma,\ -p)\,\bar{u}(\sigma,\ -p)\, e^{ip(x-x')}.\end{aligned} \tag{22.15}$$

In these equations the products of bispinors form a matrix. For instance, writing out explicitly the spinor indices, we get

$$S^{(+)}_{\alpha\beta}(x-x') = \sum_{p,\sigma} \frac{1}{2\varepsilon_p V} u_\alpha(\sigma, p)\, \bar{u}_\beta(\sigma, p)\, e^{-ip(x-x')}.$$

By direct calculation we can establish the following equations involving the bispinors $u(\sigma, p)$ and $u(\sigma, -p)$ normalized by (21.15):

$$\sum_\sigma u(\sigma, p)\, \bar{u}(\sigma, p) = \gamma p + m, \tag{22.16}$$

$$\sum_\sigma u(\sigma, -p)\, \bar{u}(\sigma, -p) = \gamma p - m. \tag{22.17}$$

These can be verified in the following manner. The right-hand side of the first equation contains only the unit matrix and the γ-matrices and, being a scalar quantity, must have the form $A\gamma p + Bm$. The left-hand side vanishes if the operator $\gamma p - m$ acts on it, so in the right-hand side A must be equal to B. Finally, in the rest frame of reference, $\mathbf{p} = 0$ and $p^0 = m$, we have (see (21.9) and (21.16)):

$$u\left(\tfrac{1}{2}\right) = \sqrt{2m}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad u\left(-\tfrac{1}{2}\right) = \sqrt{2m}\begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

whence

$$\sum_{\sigma=1/2,\,-1/2} u(\sigma, p)\, \bar{u}(\sigma, p)|_{\mathbf{p}=0} = 2m\begin{pmatrix}I & 0\\0 & 0\end{pmatrix},$$

with

$$\gamma p + m = 2m\begin{pmatrix}I & 0\\0 & 0\end{pmatrix}.$$

Consequently, $A = 1$ and the validity of (22.16) has been proved. In a similar manner we can prove (22.17).

Thus, for the positive- and negative-frequency parts of the commutation function

$$S(x - x') = \{\psi(x), \bar{\psi}(x')\}, \tag{22.18}$$

we get

$$\begin{aligned} S^{(+)}(x-x') &= \sum_{\mathbf{p}} \frac{\gamma p + m}{2\varepsilon_{\mathbf{p}} V} e^{-ip(x-x')}, \\ S^{(-)}(x-x') &= \sum_{\mathbf{p}} \frac{\gamma p - m}{2\varepsilon_{\mathbf{p}} V} e^{ip(x-x')}. \end{aligned} \tag{22.19}$$

Let us recall the definitions (20.15) and (20.17) of the functions $D_+(x)$ and $D_-(x)$ for the electromagnetic field and define similar functions for the spinor field:

$$\Delta_\pm(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2\varepsilon_p} e^{\mp ipx}. \tag{22.20}$$

Then, going over from sums to integrals in (22.19), we find that

$$\begin{aligned} S^{(+)}(x-x') &= (i\gamma\partial + m)\,\Delta_+(x-x'), \\ S^{(-)}(x-x') &= -(i\gamma\partial + m)\,\Delta_-(x-x'). \end{aligned} \tag{22.21}$$

The total commutation function

$$S(x) = S^{(+)}(x) + S^{(-)}(x) \tag{22.22}$$

is expressed in terms of the invariant function

$$\Delta_0 = i(\Delta_+ - \Delta_-) \tag{22.23}$$

in the following manner:

$$S(x) = -i(i\gamma\partial + m)\,\Delta_0(x). \tag{22.24}$$

The function $\Delta_0(x)$ differs from the electromagnetic commutation function $D_0(x)$ in that k is replaced with p in the exponential in (20.36) and k^2 with $p^2 - m^2$ in the denominator. Just as in the case with D_0, we have

$$\Delta_0(\mathbf{r}, t)\,|_{t=0} = 0,\ \frac{\partial}{\partial t}\Delta(\mathbf{r}, t)\,|_{t=0} = \delta(\mathbf{r}), \tag{22.25}$$

whence

$$S(\mathbf{r}, t)\,|_{t=0} = \gamma^0\delta(\mathbf{r}), \tag{22.26}$$

and therefore

$$\{\psi(\mathbf{r}, t), \bar{\psi}(\mathbf{r}', t')\}_{t=t'} = \gamma^0\delta(\mathbf{r} - \mathbf{r}'). \tag{22.27}$$

Function $S(x)$ constitutes a (singular) solution of the Dirac equation

$$(i\gamma\partial - m)\,S(x) = 0, \tag{22.28}$$

since

$$(\Box + m^2)\,\Delta_0(x) = 0. \tag{22.29}$$

22.3 The Causal Green Function

Let us introduce the normal and chronologically ordered products of Dirac operators in a manner similar to the one employed for electromagnetic-field operators in Section 20. However, in the case at hand a permutation of operators must result in a change in sign of the product.

The normal product of two operators is defined in such a manner that creation operators stand to the left of annihilation operators. For instance,

$$N\psi(x)\bar{\psi}(x') = N(\psi^{(+)}(x) + \psi^{(-)}(x))(\overline{\psi^{(+)}}(x') + \overline{\psi^{(-)}}(x'))$$
$$= -\overline{\psi^{(+)}}(x')\psi^{(+)}(x) + \psi^{(+)}(x)\overline{\psi^{(-)}}(x')$$
$$+ \psi^{(-)}(x)\overline{\psi^{(+)}}(x') + \psi^{(-)}(x)\overline{\psi^{(-)}}(x'), \qquad (22.30)$$

where we have allowed for the fact that $\psi^{(+)}$ and $\overline{\psi^{(-)}}$ are annihilation operators and $\psi^{(-)}$ and $\overline{\psi^{(+)}}$ are creation operators.

The chronologically ordered product of two Dirac operators $\psi_1(x)$ and $\psi_2(x')$ is defined exactly in the same manner as for electromagnetic operators, with a change in sign under permutation:

$$T\psi_1(x)\psi_2(x') = \begin{cases} \psi_1(x)\psi_2(x'), & t > t', \\ -\psi_2(x')\psi_1(x), & t' > t. \end{cases} \qquad (22.31)$$

Let us now define the contraction, or chronological pairing, of two Dirac operators:

$$\overbrace{\psi_1(x)\psi_2(x')} \equiv T\psi_1(x)\psi_2(x') - N\psi_1(x)\psi_2(x'). \qquad (22.32)$$

The reader is advised to verify that

$$\overbrace{\psi(x)\psi(x')} = \overbrace{\bar{\psi}(x)\bar{\psi}(x')} = 0. \qquad (22.33)$$

(*Hint.* The operators $\psi^{(-)}(x)$ and $\psi^{(+)}(x')$ are anticommutative for all x and x'.)

For $\psi(x)$ and $\bar{\psi}(x')$, the pairing constitutes a c-number. Indeed, let t be greater than t'. Then

$$\overbrace{\psi(x)\bar{\psi}(x')} = \psi^{(+)}(x)\overline{\psi^{(+)}}(x') + \overline{\psi^{(+)}}(x')\psi^{(+)}(x),$$

or an anticommutator, which is a c-number. A similar result is obtained when t is smaller than t'.

Since the vacuum average of an N-product is equal to zero, the vacuum average of a T-product is equal to the pairing:

$$\langle T\psi(x)\bar{\psi}(x')\rangle_0 = \overbrace{\psi(x)\bar{\psi}(x')}, \qquad (22.34)$$

which implies

$$\overbrace{\psi_\alpha(x)\bar{\psi}_\beta(x')} = \begin{cases} \langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0, & t > t', \\ -\langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0, & t < t', \end{cases} \qquad (22.35)$$

where α and β are spinor indices.

We have thus arrived at the causal function, which is commonly known as the propagator:

$$S_c(x-x') = \langle T\psi(x)\bar{\psi}(x')\rangle_0 = \overbrace{\psi(x)\bar{\psi}(x')}. \tag{22.36}$$

This function describes, for $t > t'$, the creation of an electron at point x' and the annihilation of this electron at point x, while for $t < t'$ it describes the creation of a positron at point x and the annihilation of this positron at point x'.

We can now easily express $S_c(x)$ in terms of the functions $S^{(+)}(x)$ and $S^{(-)}(x)$ introduced in (22.21). Indeed,

$$\begin{aligned} \langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle_0 &= S^{(+)}_{\alpha\beta}(x-x'), \\ \langle\bar{\psi}_\beta(x')\psi_\alpha(x)\rangle_0 &= S^{(-)}_{\alpha\beta}(x-x'), \end{aligned} \tag{22.37}$$

whereby

$$S_c(x-x') = \theta(x-x')\,S^{(+)}(x-x') - \theta(x'-x)\,S^{(-)}(x-x'). \tag{22.38}$$

The derivatives ∂_μ can be taken outside the theta functions, since ∇ is commutative with them, and $\partial_0\theta(x) = \delta(x^0)$ and $\partial_0\theta(-x) = -\delta(x^0)$, with

$$\Delta_+(x-x')|_{x_0=x_0'} = \Delta_-(x-x')|_{x_0=x_0'}.$$

Thus,

$$S_c(x-x') = (i\gamma\partial_x + m)\,\Delta_c(x-x'), \tag{22.39}$$

where

$$\Delta_c(x-x') = \theta(x-x')\,\Delta_+(x-x') + \theta(x'-x)\,\Delta_-(x-x') \tag{22.40}$$

is the propagator of a massive scalar field; the integral representation of this function is

$$\Delta_c(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2\varepsilon_p} e^{i\mathbf{p}\cdot\mathbf{r} - i\varepsilon_p|t|} = \frac{i}{(2\pi)^4}\int \frac{e^{-ipx}\,d^4p}{p^2 - m^2 + i\varepsilon},$$

$$\varepsilon_p = \sqrt{\mathbf{p}^2 + m^2}. \tag{22.41}$$

The validity of this representation can be established in the same manner as we did for the massless electromagnetic field in Section 20.

The above leads us to the following integral representation of $S_c(x)$:

$$S_c(x) = \frac{i}{(2\pi)^4}\int d^4p e^{-ipx}\frac{\gamma p + m}{p^2 - m^2 + i\varepsilon}. \tag{22.42}$$

Applying the Dirac operator to $S_c(x)$, we get

$$(i\gamma\partial - m)\,S_c(x) = i\delta(x). \tag{22.43}$$

This follows directly from the integral representation (22.42), as well as from (22.39) if we allow for the fact that

$$(\Box + m^2)\,\Delta_c(x) = -i\delta(x). \tag{22.44}$$

Thus, the propagator $S_c(x)$ is the causal Green function of the Dirac equation.

22.4 Redefining Energy and Current for the Dirac Equation

The definitions of the energy and charge of the Dirac field given in Section 21 lead, as was shown in the same section, to infinitely large negative constant terms in the eigenvalues of the Hamiltonian and the charge operator (21.35). These terms can be eliminated if both operators are redefined in an appropriate manner. For the Hamiltonian operator the definition is usually given in the form of a normal product, that is, all creation operators stand to the left of the annihilation operators, and $b_s b_s^+ \to -b_s^+ b_s$. Then instead of (21.28) we have the definition

$$H = \sum_s (\varepsilon_s^{(+)} a_s^+ a_s + \varepsilon_s^{(-)} b_s^+ b_s). \tag{22.45}$$

The current density operator is specified by a commutator instead of the usual definition:

$$j_\mu(x) = \frac{e}{2}[\bar\psi(x),\ \gamma_\mu\psi(x)] = \frac{e}{2}(\bar\psi_\alpha(\gamma_\mu)_{\alpha\beta}\psi_\beta - (\gamma_\mu)_{\alpha\beta}\psi_\beta\bar\psi_\alpha)$$

$$= \frac{e}{2}(\bar\psi\gamma_\mu\psi - \psi\gamma_\mu^T\bar\psi). \tag{22.46}$$

Then the charge, too, will rid itself of the infinite term:

$$Q = \frac{1}{2}e\int(\bar\psi\gamma^0\psi - \gamma^0\psi\bar\psi)\,d^3x = e\sum_s (N_s^{(+)} - N_s^{(-)}). \tag{22.47}$$

Obviously, if the current density operator is defined in terms of a normal product,

$$j_\mu(x) = eN\bar\psi(x)\,\gamma_\mu\psi(x), \tag{22.48}$$

we arrive at the same expression (22.47) for the charge operator. It can also be shown that not only the charge but the operators $j_\mu(x)$ given by (22.46) and (22.48) coincide. Note, finally, that j_μ can be written in charge-symmetric form. To this end we recall the definition of C-conjugation,

$$\psi^C(x) = C\bar\psi(x), \quad \bar\psi^C(x) = C^{-1}\psi(x), \tag{22.49}$$

and the definition of the C-operator: $C = -C^T$ and $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. Then from definition (22.46) follows the charge-symmetric relationship

$$j_\mu(x) = \frac{e}{2}(\bar{\psi}\gamma_\mu\psi - \bar{\psi}^C\gamma_\mu\psi^C). \tag{22.50}$$

We see that $j_\mu(x)$ is invariant under a C-transformation and a simultaneous change in sign $e \to -e$.

23. Electromagnetic Interaction

We have thus discussed two main quantized free fields, the electromagnetic and the charged Dirac. The study of the interaction of these fields constitutes the subject of quantum electrodynamics. However, there is no way in which an exact solution to the equations of quantum electrodynamics can be obtained in general form. Nevertheless, a perturbation method has been developed, which, thanks to the smallness of the coupling constant of the electromagnetic interaction, or the fine structure constant $\alpha = e^2/\hbar c = 1/137$, makes it possible to find an asymptotic solution for practically any interaction between electrons and photons. But before we discuss this method, let us formulate the exact equations of quantum electrodynamics.

23.1 The Heisenberg Field Equations

Let us start with two interacting fields, the electron-positron field and the electromagnetic field. To each field we assign an operator in the Heisenberg representation,

$$\psi(x), \quad \bar{\psi}(x), \quad A_\mu(x), \tag{23.1}$$

which depends explicitly not only on r but on time t as well. For these operators we can write the same equations as for classical interacting fields:

$$(i\gamma\partial - e\gamma A(x) - m)\psi(x) = 0, \quad \bar{\psi}(x)(i\gamma\partial + e\gamma A(x) + m) = 0, \tag{23.2a}$$

$$\Box A_\mu(x) = 4\pi j_\mu(x), \tag{23.2b}$$

where the 4-current density can be written in the charge-symmetric form,

$$j_\mu(x) = \frac{e}{2}[\bar{\psi}(x), \quad \gamma_\mu\psi(x)] = \frac{e}{2}(\bar{\psi}(x)\gamma_\mu\psi(x) - \bar{\psi}^C(x)\gamma_\mu\psi^C(x)). \tag{23.3}$$

These equations can be obtained by employing the variational principle for the action integral of the Lagrangian density. It can be

proved, in general form, that Eqs. (23.2) are gauge invariant and that the total 4-momentum of the fields and charges is conserved, which follows from the Lorentz invariance and gauge invariance of the theory. We will not do this here for the general case, but in what follows we will often use these results and in the next chapter we will demonstrate their validity using concrete examples.

The equations (23.2a) and (23.2b) for the interacting fields are nonlinear and inhomogeneous. Therefore, we cannot solve them in general form, that is, write out the explicit dependence of the operator functions ψ, $\bar{\psi}$, and A on x, in contrast to the case of free fields.

For the same reason it is impossible to write out the commutation relations for these fields at arbitrary moments of time, since we do not know the time dependence of the fields. However, for coincident times such commutation relations can be written; these are known as equal-time commutation relations. It is expedient to fix these relations in such a manner that they coincide with the commutation relations for free fields:

$$\begin{aligned} \left[\mathrm{A}_\mu(x),\ \frac{\partial \mathrm{A}_\nu}{\partial t}(x')\right]_{t=t'} &= -4\pi i g_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}'), \\ [\mathrm{A}_\mu(x),\ \mathrm{A}_\nu(x')]_{t=t'} &= 0, \\ \{\psi_\alpha(x),\ \psi^+_\beta(x')\}_{t=t'} &= \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'). \end{aligned} \tag{23.4}$$

All the other commutation relations are equal to zero at $t = t'$:

$$\{\psi,\ \psi\}_{t=t'} = \{\psi^+,\ \psi^+\}_{t=t'} = 0,$$
$$[\psi,\ \mathrm{A}]_{t=t'} = [\psi,\ \partial \mathrm{A}/\partial t']_{t=t'} = 0,$$

etc. As for the state vectors $|\Phi\rangle$ for a system of fields, in the Heisenberg representation they are time independent:

$$\frac{\partial}{\partial t}|\Phi\rangle = 0. \tag{23.5}$$

As noted earlier, the electromagnetic current is conserved:

$$\partial \mathrm{j}_\mu/\partial x^\mu = 0. \tag{23.6}$$

Therefore, calculating the 4-divergence of the right- and left-hand sides of Eq. (23.2b) for the vector potential A_μ, we get

$$\frac{\partial}{\partial x_\mu}\Box \mathrm{A}_\mu = \Box \frac{\partial \mathrm{A}_\mu}{\partial x_\mu} = 4\pi \frac{\partial \mathrm{j}_\mu}{\partial x_\mu} = 0,$$

that is, the divergence of A_μ satisfies homogeneous d'Alembert's equation

$$\Box \mathrm{A}^\mu_{,\mu} = 0. \tag{23.7}$$

This means that $A^{\mu}_{,\mu}$ can be decomposed into negative- and positive-frequency parts, as it was done for free fields:

$$A^{\mu}_{,\mu} = A^{\mu(+)}_{,\mu} + A^{\mu(-)}_{,\mu}. \tag{23.8}$$

Hence, we can impose the Lorentz condition on the state vectors:

$$A^{\mu(+)}_{,\mu} \mid \Phi\rangle = 0, \quad \langle \Phi \mid A^{\mu(-)}_{,\mu} = 0. \tag{23.9}$$

Thus, the wave equation (23.2b) leads to the Maxwell equations in the mean:

$$\left\langle \frac{\partial F_{\mu\nu}}{\partial x_\nu} \right\rangle = -4\pi \langle j_\mu \rangle. \tag{23.10}$$

Let us write the Hamiltonian operator for the Dirac field:

$$H_D = \int d^3x \frac{i}{2} (\bar{\psi}\gamma^0\psi_{,t} - \bar{\psi}_{,t}\gamma^0\psi) = \int d^3x \frac{i}{2} (\bar{\psi}\gamma^0\psi_{,t} + \bar{\psi}^C\gamma^0\psi^C_{,t}). \tag{23.11}$$

This expression corresponds to the classical definition of H given in Chapter I, and the charge-symmetric form is obtained in the same manner as we did for the 4-current density in Section 22. Excluding the time derivatives, that is, going over to canonical variables via Dirac equations,

$$i\gamma^0\psi_{,t} = (e\gamma A + m)\psi - i\gamma\nabla\psi,$$
$$i\gamma^0\psi^C_{,t} = (-e\gamma A + m)\psi^C - i\gamma\nabla\psi^C$$

(note that the charge enters the equation for ψ^C with the opposite sign), we obtain

$$H_D = \int \left[\frac{1}{2}\bar{\psi}(-i\gamma\nabla + m)\psi + \frac{1}{2}\bar{\psi}^C(-i\gamma\nabla + m)\psi^C \right.$$
$$\left. + \frac{e}{2}(\bar{\psi}\gamma^\mu\psi - \bar{\psi}^C\gamma^\mu\psi^C) A_\mu \right] d^3x = H^0_D + H_{int}, \tag{23.12}$$

where

$$H_{int} = \int d^3x j_\mu A^\mu = \frac{e}{2} \int d^3x (\bar{\psi}\gamma^\mu\psi - \bar{\psi}^C\gamma^\mu\psi^C) A_\mu \tag{23.13}$$

is the so-called interaction Hamiltonian, and H^0_D is the free Dirac Hamiltonian.

If we also take into account the free Hamiltonian of the electromagnetic field, H^0_f, we will arrive at the total Hamiltonian for a system of interacting electron-positron and electromagnetic fields:

$$H = H^0_f + H^0_D + H_{int} = H^0 + H_{int}, \tag{23.14}$$

where H^0 does not depend on the products of components of the various fields, that is, is the free Hamiltonian. Since H_{int} defined by (23.13) depends on the product of three cofactors corresponding to

the charged Dirac and electromagnetic fields, it is called the interaction Hamiltonian. It can easily be verified, although we will not do this here, that the field equations (23.2a) and (23.2b) are Heisenberg equations of motion for the operators ψ, $\bar{\psi}$, and A and may be obtained from the general equation for Heisenberg operators,

$$\frac{du(x)}{dt} = i[H, u(x)], \tag{23.15}$$

in which we must put $u(x) = \psi(x)$, $\bar{\psi}(x)$, and $A(x)$, and for H we must take (23.14). We must also use equal-time commutation relations (23.4).

23.2 The Interaction Representation (Dirac Picture)

For an approximate solution of the system of equations for interacting fields it often proves convenient to transfer from the Heisenberg representation to the interaction representation, in which the operators obey the free equations. This representation is also known as the Dirac picture. In the Schrödinger representation the state vectors obey the Schrödinger equation

$$i\frac{\partial}{\partial t}|\Phi(t)\rangle = H|\Phi(t)\rangle, \tag{23.16}$$

where $H = H_0 + H_{int}$, and the operators u (r) are time independent.

Let us transfer to the interaction representation, assuming that the operators $u_I(x)$ and the state vectors $|\Phi_I(t)\rangle$ in this representation are

$$\begin{aligned} u_I(x) &= e^{iH_0t}u(r)e^{-iH_0t}, \\ |\Phi_I(t)\rangle &= e^{iH_0t}|\Phi(t)\rangle. \end{aligned} \tag{23.17}$$

In accord with this definition, the operators $u_I(x)$ satisfy the equation

$$\frac{du_I}{dt} = i[H_0, u_I], \tag{23.18}$$

while state vectors, by virtue of (23.16), obey the equation

$$i\frac{\partial}{\partial t}|\Phi_I(t)\rangle = H_I(t)|\Phi_I(t)\rangle, \tag{23.19}$$

where

$$H_I = e^{iH_0t}H_{int}e^{-iH_0t} \tag{23.20}$$

is the interaction operator, that is, the interaction Hamiltonian in the Dirac picture.

The reader will recall that the Heisenberg and Schrödinger operators and the state vectors are related through the following formulas ($\partial H/\partial t = 0$):

$$u(x) = e^{iHt} u(r) e^{-iHt}, \tag{23.21a}$$

$$|\Phi(t)\rangle = e^{-iHt} |\Phi\rangle, \tag{23.21b}$$

with the Heisenberg operators obeying the equation

$$\frac{du}{dt} = i[H, u], \tag{23.22}$$

and the Heisenberg state vectors being time independent,

$$\frac{\partial}{\partial t} |\Phi\rangle = 0. \tag{23.23}$$

We see that the interaction representation is an intermediate, so to say, between the Schrödinger and Heisenberg representations: the operators satisfy the Heisenberg equation of motion with the free Hamiltonian, while the state vectors obey the Schrödinger equation with the interaction operator as the Hamiltonian.

Equation (23.21b) constitutes a formal solution to the Schrödinger equation for the state vector $|\Phi(t)\rangle$ and can be rewritten thus:

$$|\Phi(t)\rangle = U(t) |\Phi\rangle, \tag{23.24}$$

where

$$U(t) = e^{-iHt} \tag{23.25}$$

is the evolution operator, which connects the state vectors $|\Phi\rangle$ at $t = 0$ and $|\Phi(t)\rangle$ at $t \neq 0$. Obviously, for arbitrary times t' and t'' $(t'' > t')$ we get

$$|\Phi(t'')\rangle = U(t'' - t') |\Phi(t')\rangle, \tag{23.26}$$

where

$$U(t'' - t') = e^{-iH(t''-t')}. \tag{23.27}$$

Formula (23.26) is no more than a formal solution to the Schrödinger equation since we know neither how operator H acts on the state vectors nor the commutation relations involving the operators that constitute H. We will therefore go over in (23.26) to the interaction representation via (23.17):

$$|\Phi_I(t'')\rangle = e^{iH_0 t''} U(t'' - t') e^{-iH_0 t'} |\Phi_I(t')\rangle = U_I(t'', t') |\Phi_I(t')\rangle, \tag{23.28}$$

where

$$U_I(t'', t') = e^{iH_0 t''} U(t'' - t') e^{-iH_0 t'} \tag{23.29}$$

is the evolution operator in the interaction representation. This equation is again formal, however. It does not enable us to determine

the explicit form of the evolution operator, since it expresses this operator in terms of unknown operators u (r) in the Schrödinger representation, which operators are incorporated in H and H_0. Note, however, that the evolution operator (23.29) satisfies the same equation (23.19) as the state vector in the Dirac picture does,

$$i\frac{\partial}{\partial t}U_I(t, t') = H_I(t)\,U_I(t, t'). \tag{23.30}$$

Whence, to find the evolution operator in explicit form, we must solve this equation. Note that the operator $H_I(t)$ here depends on time explicitly, and therefore we are unable to solve the nonlinear operator equation exactly, as we did with the Schrödinger equation (23.16) when we had to find operator (23.27). At the same time, Eq. (23.30) has an important advantage that justifies the transfer to the interaction representation. The fact is that all the operators in (23.20) are given in the interaction representation. The explicit form of operator $H_I(t)$, obviously, is the same as H_{int}, only it is expressed in terms of field operators in the interaction representation instead of in terms of Heisenberg fields. The same is true of the free operator H_0 expressed in terms of fields in the interaction representation. Thus, from (23.18) we obtain the ordinary free equations for the field operators:

$$(i\gamma\partial - m)\,\psi(x) = 0,\quad \bar{\psi}(x)\,(i\gamma\partial + m) = 0,\quad \Box A_\mu(x) = 0. \tag{23.31}$$

Moreover, the equal-time commutation relations for ψ, $\bar{\psi}$, A_μ and $\pi_\mu = -(1/4\pi)\,\partial A_\mu/\partial t$, obviously, are the same as in the Heisenberg representation, (23.4). On the basis of the equal-time commutation relations (23.4) and the solutions to Eqs. (23.31) one can obtain non-equal-time ($t \neq t'$) commutation relations that coincide with those for free fields (see Sections 20 and 22).

Thus, the theory of free fields developed in the preceding sections can be applied to describe the evolution of operators in the Dirac picture. At the same time, the equation of motion for the state vector, (23.19), or for the evolution operator, (23.30), contains only the interaction operator and in cases when the interaction is weak can be solved approximately.

Instead of Eq. (23.30), let us take the equivalent integral equation

$$U_I(t, t') = 1 - i\int_{t'}^{t} H_I(t_1)\,U_I(t_1, t')\,dt_1. \tag{23.32}$$

Successive iterations lead to the following series:

$$U_I(t, t') = 1 - i\int_{t'}^{t} H_I(t_1)\,dt_1 + (-i)^2\int_{t'}^{t} dt_2\,H_I(t_2)\int_{t'}^{t_2} dt_1 H_I(t_1) +$$

$$\ldots + (-i)^n \int_{t'}^{t} dt_n \int_{t'}^{t_n} dt_{n-1} \ldots \int_{t'}^{t_2} dt_1 \mathrm{H_I}(t_n)\, \mathrm{H_I}(t_{n-1}) \ldots \mathrm{H_I}(t_1) + \ldots, \tag{23.33}$$

where the operators $\mathrm{H_I}(t_1)$, $\mathrm{H_I}(t_2)$, etc. are not commutative.

In expansion (23.33),

$$t' < t_1 < t_2 < \ldots < t_{n-1} < t_n < t \tag{23.34}$$

for any $n = 1, 2, 3, \ldots$. In each term of the series the operators follow in chronological order. We can write the series in compact form thus:

$$\mathrm{U_I}(t'', t') = T \exp\left(-i \int_{t'}^{t''} \mathrm{H_I}(t)\, dt\right), \tag{23.35}$$

and this series is known as the T-exponential. If the interaction is weak, we can take a finite number of terms in series (23.33) (or "truncate" the series). The matrix elements of the evolution operators in this approximation calculated between free field states are the transition amplitudes in the respective perturbation-theory order. Electrodynamics proved to be the only theory for which such an expansion is justified and provides excellent results. This is due to the relatively simple Abelian structure of electrodynamics and the smallness of the coupling constant, the fine structure constant $\alpha \approx 1/137$.

24. The S-matrix for Electromagnetic Interaction

The solution of the equation of motion for the evolution operator in the form of a T-exponential obtained in the previous section makes it possible, at least in principle, to describe electromagnetic processes in any perturbation-theory order in the fine structure constant. This requires, however, a refinement of the statement of the problem, which statement differs considerably from the well-known nonrelativistic perturbation theory used, for example, in scattering theory.

24.1 Dyson's Formula

In electrodynamics the interaction operator has the form

$$\mathrm{H_I}(t) = \int d^3x \mathrm{j}_\mu(x) \mathrm{A}^\mu(x), \tag{24.1}$$

where the current and electromagnetic-potential operators are given in the interaction representation, with charge e incorporated in the definition of current (23.3).

Let us consider a statement of the problem in which the particles are observed only in the asymptotic states, that is, long before and after the interaction. In other words, we observe the states of particles at $t' \to -\infty$ and $t'' \to +\infty$. This means we must calculate the appropriate limit of the evolution operator,

$$S = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} e^{iH_0 t''} e^{-iH(t''-t')} e^{-iH_0 t'} = \lim_{\substack{t' \to -\infty \\ t'' \to +\infty}} T \exp\left(-i \int_{t'}^{t''} H_I(t)\, dt\right), \tag{24.2}$$

which limit is known as the S-matrix. We must also assume that the interaction is switched off in some manner in these asymptotic states, that is,

$$\lim_{t \to \pm\infty} H_I(t) = 0. \tag{24.3}$$

Let us assume, at the same time, that the sets of states of the system of particles as $t \to +\infty$ and $t \to -\infty$, that is, the vectors of asymptotic states in the interaction representation,

$$|\Phi_s(+\infty)\rangle = |\Phi_s^{(+)}\rangle \text{ and } |\Phi_s(-\infty)\rangle = |\Phi_s^{(-)}\rangle,$$

coincide with the eigenvectors of the free Hamiltonian.

The described situation seems, at least at first glance, to be completely equivalent to the problem of potential scattering in nonrelativistic quantum mechanics. This is not the case, however. The fact is that even for asymptotic states with $t \to \pm\infty$ we cannot exclude the interaction completely. Although real particles prove to be far apart in such states, so far that their interaction with each other can be ignored, there still remains the interaction with the so-called vacuum, that is, with virtual, or to-be-created, particles. This last interaction leads, for instance, to a change in the electron mass and to renormalization of the electron charge, just like an electric charge of a particle placed in a medium is screened thanks to the polarization of the medium.

To allow for these effects it is common practice to assume that at the beginning the interaction is switched on very slowly, so that the noninteracting particles, which possess bare masses and charges, are transformed into the "dressed", real, particles that are in the same states $|\Phi_s\rangle$ with the same quantum numbers as the initial "bare" particles but possess physically observable masses and charges. This assumption is known as the adiabatic hypothesis (e.g. see [7]). The dressed particles then scatter because of the interaction or change their state in some other manner. For the time being we will not touch on the subject of how particles acquire physical charges and masses—this will be left to the final chapter of the book. In accord with what we have said above, let us assume that the system

of interacting fields as $t \to -\infty$ is in a certain initial stationary state described by an eigenvector $|\Phi^{(-)}\rangle = |\Phi_i\rangle$ of the free Hamiltonian H_0 with physically meaningful masses and charges. The final state

$$|\Phi^{(+)}\rangle = \sum_f c_f |\Phi_f\rangle = S|\Phi^{(-)}\rangle \tag{24.4}$$

is a series expansion in the initial set of eigenvectors of H_0 with physically meaningful masses and charges. The expansion coefficients are the probability amplitudes of the transition of the system from state $|\Phi_i\rangle$ to one of the possible states $|\Phi_f\rangle$:

$$c_f = \langle f | S | i \rangle = S_{fi}. \tag{24.5}$$

We see that these amplitudes coincide with the matrix elements of the S-matrix: $S_{fi} = \langle f | S | i \rangle$. According to the solution (23.33) for the evolution operator, we arrive at the following expansion for the S-matrix:

$$\begin{gathered} S = \sum_{n=0}^{\infty} S_n, \quad S_0 = 1 \quad (n=0), \\ S_n = (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n \, H_I(t_1) H_I(t_2) \ldots H_I(t_n) \\ (n \neq 0). \end{gathered} \tag{24.6}$$

In n-dimensional space, the domain of integration for the nth term of the series is restricted by the conditions

$$-\infty < t_n \leqslant t_{n-1} \leqslant \ldots \leqslant t_2 \leqslant t_1 < \infty, \tag{24.7}$$

where in the integrands the interaction operators are positioned from right to left in the order of increasing time, that is, in chronological order.

Let us take the term in the series with $n = 2$. It is clear that the result will be the same if we redesignate the integration variables, $t_2 \to t_1$, $t_1 \to t_2$:

$$\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \, H_I(t_1) H_I(t_2) = \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 H_I(t_2) H_I(t_1). \tag{24.8}$$

Adding the right term to the left, we get

$$\begin{aligned} &\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1) H_I(t_2) + \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 H_I(t_2) H_I(t_1) \\ &\qquad = \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 T H_I(t_1) H_I(t_2), \end{aligned} \tag{24.9}$$

or, by virtue of (24.8),

$$\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \mathrm{H_I}(t_1)\,\mathrm{H_I}(t_2) = \frac{1}{2}\int_{-\infty}^{\infty} dt_1 \int_{\infty}^{\infty} dt_2 T\mathrm{H_I}(t_1)\,\mathrm{H_I}(t_2). \quad (24.10)$$

We can rewrite any other term in (24.6) in exactly the same manner by carrying out a permutation of variables $t_1, \ldots, t_n \to t_{i_1}, t_{i_2}, \ldots, t_{i_n}$. Adding all such integrals, which are equal to each other, we get

$$(-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ldots \int_{-\infty}^{\infty} dt_n T\mathrm{H_I}(t_1)\,\mathrm{H_I}(t_2)\ldots\mathrm{H_I}(t_n). \quad (24.11)$$

Since the number of such permutations is $n!$, we can find the value of S_n by dividing (24.11) by $n!$:

$$\mathrm{S}_n = \frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ldots \int_{-\infty}^{\infty} dt_n T\mathrm{H_I}(t_1)\,\mathrm{H_I}(t_2)\ldots\mathrm{H_I}(t_n). \quad (24.12)$$

The S-matrix expansion (24.6) in which the separate terms are defined via (24.12) can be symbolically represented in the form of a T-exponential (Dyson's formula):

$$\mathrm{S} = T\exp\left(-i\int_{-\infty}^{\infty} dt\mathrm{H_I}(t)\right), \quad (24.13)$$

where the electromagnetic-interaction operator is

$$\mathrm{H_I}(t) = \int d^3x \mathrm{j}_\mu(x)\mathrm{A}^\mu(x) = e\int d^3x N\bar{\psi}(x)\gamma_\mu \mathrm{A}^\mu(x)\psi(x), \quad (24.14)$$

the field operators are taken in the interaction representation, and the electromagnetic current is defined via the normal product.

24.2 Wick's Theorem

The probabilities of transitions induced by an interaction are expressed in terms of elements of the S-matrix between free physical states, (24.5). The representation (24.12) of the terms of the S-matrix is inconvenient for calculating these matrix elements.

Indeed, for real calculations we arrive at quantities of the form

$$\langle f \mid T\mathrm{u}_1\mathrm{u}_2 \ldots \mathrm{u}_n \mid i\rangle, \quad (24.15)$$

where u_k ($k = 1, 2, \ldots, n$) are field operators, and the initial and final states,

$$|\, t\rangle = A_I^{(-)}(-\infty)\,|\,0\rangle,\quad \langle f\,| = \langle 0\,|\,B_I^{(+)}(\infty) \tag{24.16}$$

are expressed in terms of creation and annihilation operators in the Dirac picture at $t = -\infty$ and $t = +\infty$, respectively, that is, $A_I^{(-)}(-\infty)$ and $B_I^{(+)}(\infty)$, as acting on the physical vacuum $|\,0\rangle$. For this reason, it would be more convenient if we could represent S_n in such a form that instead of the T-product the operators would be under the sign of an N-product, that is, all creation operators would be placed to the left of all annihilation operators. Then, if the number of annihilation operators for particles of a definite type in the N-product is equal to the number of the corresponding creation operators in $A_I(-\infty)$ we finally obtain, by interchanging them a necessary number of times and applying them to the vacuum state, a c-number as the factor of the vacuum state $|\,0\rangle$. A similar procedure with the $\langle f\,|$ state will lead to a c-number as the factor of $\langle 0\,|$. These factors, obviously, consist of various commutation functions of the field operators. As a result we get rid of all the operators and, in the end, arrive at the sought transition amplitude.

The procedure described above requires first of all the reduction of the S-matrix to normal form. This is carried out via Wick's theorem, which states that any chronologically ordered product is equal to the sum of all possible normal products that can be formed with all possible pairings.

Symbolically, the essence of the theorem can be formulated by the following equation:

$$T u_1 u_2 \ldots u_n = N u_1 u_2 \ldots u_n + \sum N \overbrace{u_1 u_2} \ldots u_n + \sum \overbrace{u_1 u_2 u_3 \ldots u_n}, \tag{24.17}$$

where the summation signs stand for summation over all possible pairings of a definite number of pairs of operators (only two operators are paired in the second term, while in the last term the maximum number of operators are paired), with the operators, taken in the interaction representation, being the free field operators.

For $n = 2$, Wick's theorem has actually been proved, since according to the definition of chronological pairing, or contraction, of two operators (see Sections 20 and 22), we get

$$T u_1 u_2 = N u_1 u_2 + \overbrace{u_1 u_2}. \tag{24.18}$$

Suppose that Wick's theorem is valid for an arbitrary n. Let us prove that it is valid for an $(n + 1)$st operator under the sign of the T-product. Since all the u_k can be decomposed into creation and an-

nihilation operators, we assume that the u_k are either creation or annihilation operators. Let us now rearrange the operators on the right and left of (24.17) so that they stand in chronological order. The sign factors may change if we are dealing with fermion operators, but this change will be the same on both sides and (24.17) will not be violated. Thus, let us assume that

$$t_1 > t_2 > \ldots > t_n, \tag{24.19}$$

whereby

$$T u_1 u_2 \ldots u_n = u_1 u_2 \ldots u_n. \tag{24.20}$$

Suppose u_{n+1} is a creation operator, that is, $u_{n+1} = u^{(-)}_{n+1}$, with $t_n > t_{n+1}$. Then

$$T u_1 \ldots u_n u^{(-)}_{n+1} = T(u_1 \ldots u_n) u^{(-)}_{n+1}. \tag{24.21}$$

For the T-product on the right-hand side we have expansion (24.17). We will interchange $u^{(-)}_{n+1}$ with all the $u^{(+)}_k$ moving to the left until we meet $u^{(-)}_k$ operators. As a result of such permutations the following pairings occur:

$$u^{(+)}_k u^{(-)}_{n+1} = u^{(-)}_{n+1} u^{(+)}_k + \overbrace{u^{(+)}_k u^{(-)}_{n+1}}, \tag{24.22}$$

since the operators were chronologically ordered, that is, $t_k > t_{n+1}$. The operator $u^{(-)}_{n+1}$ is commutative (anticommutative) with all the $u^{(-)}_k$. As a result we obtain

$$\pm u^{(-)}_{n+1} T u_1 u_2 \ldots u_n, \tag{24.23}$$

and also the sum of terms with various pairings of $u^{(-)}_{n+1}$ and $u^{(+)}_k$ of the form

$$\pm \overbrace{u^{(+)}_k u^{(-)}_{n+1}} T u_1 u_2 \ldots u_{k-1} u_{k+1} \ldots u_n. \tag{24.24}$$

The sign in front of the terms in (24.23) and (24.24) corresponds to the parity of the number of permutations of fermion operators. Interchanging the order of these fermion operators under the sign of the N-product so as to restore chronological order, we obtain a "plus" sign.

Wick's theorem is valid for the T-product of the product of $n - 1$ operators in (24.24) because it is valid for n operator cofactors. The remaining pairings, that is, those that are not present in (24.24), are all zero,

$$\overbrace{u^{(-)}_k u^{(-)}_{n+1}} = 0,$$

since operators that are either both positive-frequency or both negative-frequency are commutative, or, in the case of fermions, are anticommutative. Thus, we can assume that the expansion contains

all possible pairings, which means that Wick's theorem has been proved for the case at hand.

If time t_{n+1} is not the earliest possible, as it was assumed, but lies somewhere between t_1 and t_n, say, $t_k > t_{n+1} > t_{k+1}$, all that we have just proved concerning operators $u_1, u_2, \ldots, u_k$ remains valid, while the pairings of $u^{(-)}_{n+1}$ with the "earlier" operators, say, u_{k+l} are zero, since

$$\overbrace{u^{(-)}_{n+1} u^{(-)}_{k+l}} \equiv 0, \quad \overbrace{u^{(-)}_{n+1} u^{(+)}_{k+l}} = 0 \qquad (t_{n+1} > t_{k+l}).$$

Similar reasoning can be carried out for the case where $u_{n+1} = u^{(+)}_{n+1}$. Thus, Wick's theorem has been proved for the general case as well.

Note that if under the T-product sign there are N-products (a mixed T-product), the operators under the N-product sign would not be rearranged and there would be no pairings between them in the expansion. If we bear in mind this remark, Wick's theorem is valid for mixed T-products, too.

25. Radiation Emitted by a Classical Current

There are only a few problems in quantum electrodynamics for which exact solutions can be found. These are model problems in which certain conditions or restrictions that considerably simplify the problem are made. One of such problems deals with finding the quantized electromagnetic field generated by an external classical source, the 4-current density $j_\mu(x)$, which constitutes a known function of position and time.

25.1 The Electromagnetic Field of a Classical Source

We will obtain the operator that represents the interaction of the classical source with the quantum field from the general formula (24.1) in which the 4-current density operator is replaced with the classical function

$$H_I = \int d^3x j_\mu(x) A^\mu(x). \tag{25.1}$$

In essence, the term "interaction" does not reflect the true picture, since here we are speaking most likely of how a given source acts on a dynamical system, the quantized electromagnetic field, without reaction from that system, that is, the system splits and ceases to be self-consistent. The S-matrix in this case will be simpler than (24.12),

since all classical currents j_μ can be taken outside the T-product sign:

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\ldots d^4x_n j_\mu(x_1)\ldots j_\nu(x_n)\,TA^\mu(x_1)\ldots A^\nu(x_n). \tag{25.2}$$

According to Section 20, for chronological pairing we have

$$\overbrace{A_\mu(x_1)\,A_\nu(x_2)}=-4\pi g_{\mu\nu}D_c(x_1-x_2)=-iD_{\mu\nu}(x_1-x_2), \tag{25.3}$$

where instead of the causal propagator $D_c(x_1-x_2)$ we have introduced the function

$$D_{\mu\nu}(x)=g_{\mu\nu}D(x),\quad D(x)=-4\pi iD_c(x). \tag{25.4}$$

Now, using Wick's theorem, we write out the T-product explicitly:

$$TA(1)\ldots A(n)=\sum(-i)^k D(12)\ldots D(2k-1,\,2k)\,A^{(-)}(2k+1)\ldots A^{(-)}(2k+r)\,A^{(+)}(2k+r+1)\ldots A^{(+)}(n). \tag{25.5}$$

For the sake of simplicity we did not write out the vector indices and instead of the arguments x_k we simply wrote their numbers, say, $D(12)\equiv D(x_1-x_2)$. Summation is carried out over all values of k and r that satisfy the condition $2k+r\leqslant n$. For example, for $n=2$ with $t_1>t_2$ we get

$$\begin{aligned}TA(1)\,A(2)&=A^{(+)}(1)\,A^{(+)}(2)+A^{(-)}(1)\,A^{(+)}(2)+A^{(+)}(1)\,A^{(-)}(2)\\&\quad+A^{(-)}(1)\,A^{(-)}(2)=A^{(+)}(1)\,A^{(+)}(2)+A^{(-)}(1)\,A^{(+)}(2)\\&\quad+A^{(-)}(1)\,A^{(-)}(2)+A^{(-)}(2)\,A^{(+)}(1)+\overbrace{A^{(+)}(1)\,A^{(-)}(2)}.\end{aligned} \tag{25.6}$$

For the first four terms we have $k=0$, $r=0,\,1,\,1,\,2$, while for the last term we have $k=1$, $r=0$.

Upon integrating the S-matrix expansion (25.2) at fixed k and r, we arrive at identical terms that differ only in the permutation of the arguments. In all there are $n!$ permutations some of which are not admissible, namely, 2^k permutations of the arguments of the D-functions, $k!$ permutations of the D-functions, $r!$ permutations of the $A^{(-)}$ operators, and $(n-2k-r)!$ permutations of the $A^{(+)}$ operators prove to be excluded. The example of (25.6) illustrates this combinatorics fairly well. Thus, in all there are

$$\frac{n!}{2^k k!r!\,(n-2k-r)!}$$

identical terms, and the S-matrix expansion assumes the form

$$S=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\sum_{r=0}^{n-2k}\frac{(-i)^{n+k}}{2^k k!\,r!\,(n-2k-r)!}\int d^4x_1\ldots d^4x_n\prod_{s=1}^{n}j(s)$$
$$\times\prod_{t=1}^{k}D(2t-1,2t)\prod_{u=2k+1}^{2k+r}A^{(-)}(u)\prod_{v=2k+r+1}^{n}A^{(+)}(v). \qquad (25.7)$$

25.2 Soft-Photon Emission

Let us calculate the transition amplitude from the vacuum state $|0\rangle$ to the state with N photons, $|N\rangle$, which amplitude corresponds to N-photon emission by a classical current. Then a nonzero contribution will be provided only by those terms in the S-matrix expansion that contain no photon annihilation operator $A^{(+)}$ but contain N creation operators $A^{(-)}$ (i.e. $r=N=n-2k$), with the result that

$$\langle N|S|0\rangle=\frac{(-i)^N}{N!}\int dx_1\ldots dx_N\langle N|\prod_{i=1}^{N}j(i)A^{(-)}(i)|0\rangle$$
$$\times\sum_{k=0}^{\infty}\frac{i^k}{2^k k!}\int dx_1\ldots dx_{2k}j(1)\ldots j(2k)D(12)\ldots D(2k-1,2k). \qquad (25.8)$$

Although we do not write out the vector indices explicitly, it is understood that, for example (see (25.4)),

$$j(1)D(12)j(2)=j_\mu(1)D^{\mu\nu}(12)j_\nu(2)=j_\mu(1)D(12)j^\mu(2). \qquad (25.9)$$

In the expansion in k in (25.8) we can easily see a series expansion for the exponential:

$$\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{i}{2}\int d^4x_1\,d^4x_2 j(1)D(12)j(2)\right]^k=e^{-\nu/2}, \qquad (25.10)$$

where

$$\nu=-i\int d^4x_1\,d^4x_2 j(1)D(12)j(2)=\nu_1+i\nu_2 \qquad (25.11)$$

is a complex-valued quantity, with ν_1 and ν_2 the real and imaginary parts of the quantity.

Then instead of (25.8) we obtain

$$\langle N \mid \mathrm{S} \mid 0\rangle = \frac{(-i)^N}{N!} e^{-\nu/2} \int d^4x_1 \ldots d^4x_N \langle N \mid \prod_{i=1}^{N} j(i) \mathrm{A}^{(-)}(i) \mid 0\rangle. \tag{25.12}$$

The probability of N transversal photons with all possible momenta and with polarizations $\lambda = 1, 2$ being created is determined by the square of the modulus of amplitude (25.12):

$$P(N) = \sum_{N_{\mathbf{k}\lambda}\,(\lambda=1,\,2)} |\langle N_{\mathbf{k}\lambda} \mid \mathrm{S} \mid 0\rangle|^2, \tag{25.13}$$

where summation is carried out over all sets of occupation numbers $N_{\mathbf{k}\lambda}$ for all values of momenta and transverse polarizations $\lambda = 1, 2$ restricted by the condition

$$\sum_{\mathbf{k}} \sum_{\lambda=1,\,2} N_{\mathbf{k}\lambda} = N. \tag{25.14}$$

Substituting the emission amplitude of N real photons, (25.12), into (25.13), we arrive at the sum

$$\sum_{N_{\mathbf{k}\lambda}\,(\lambda=1,\,2)} \langle 0 \mid \mathrm{A}^{(+)}(1) \ldots \mathrm{A}^{(+)}(N) \mid N_{\mathbf{k}\lambda}\rangle \langle N_{\mathbf{k}\lambda} \mid \mathrm{A}^{(-)}(N+1) \ldots \mathrm{A}^{(-)}(2N) \mid 0\rangle, \tag{25.15}$$

with condition (25.14) implied.

Moreover, we note that formula (25.12) is gauge invariant, that is, permits transformations of polarization vectors (see (19.44)) of the form

$$e_\mu \to e'_\mu = e_\mu + f k_\mu, \tag{25.16}$$

under which the amplitude does not change. This type of transformation results in the photons acquiring longitudinal and temporal components in the polarization vectors, which are linked through the Lorentz condition (19.43)

$$e_\mu k^\mu = e'_\mu k^\mu = 0. \tag{25.17}$$

Thus, among the states $\mid N_{\mathbf{k}\lambda}\rangle$ we may find states with longitudinal and temporal photons, states that, as we know, cannot be observed and whose number, therefore, is not restricted by condition (25.14).

Thus, summation in (25.13) can be extended to all kinds of states with arbitrary occupation numbers $N_{\mathbf{k}\lambda}$:

$$0 \leqslant \sum_{\mathbf{k}\lambda=1,\,2,\,3,\,0} N_{\mathbf{k}\lambda} < \infty. \tag{25.18}$$

The system of such states is complete, that is,

$$\sum_{N_{k\lambda}}{}' | N_{k\lambda}\rangle \langle N_{k\lambda} | = \mathrm{I}, \tag{25.19}$$

where summation is extended to all values of $N_{k\lambda}$, (25.18), for all $\mathbf{k}$ and $\lambda = 1, 2, 3, 0$.

Thus, using the property of gauge invariance, extending the summation in (25.15) to all $N_{k\lambda}$ ($\lambda = 1, 2, 3, 0$), and employing the completeness condition (25.19), we get

$$\sum_{N_{k\lambda}} \langle 0 | \mathrm{A}^{(+)}(1) \ldots \mathrm{A}^{(+)}(N) | N_{k\lambda}\rangle \langle N_{k\lambda} | \mathrm{A}^{(-)}(N+1) \ldots \mathrm{A}^{(-)}(2N) | 0\rangle$$

$$= \langle 0 | \prod_{i=1}^{N} \mathrm{A}^{(+)}(i) \prod_{i'=N+1}^{2N} \mathrm{A}^{(-)}(i') | 0\rangle. \tag{25.20}$$

Then probability (25.13) assumes the form

$$P(N) = \frac{e^{-\nu_1}}{(N!)^2} \int dx_1 \ldots dx_N\, dx_{N+1} \ldots dx_{2N}$$

$$\times \langle 0 | \prod_{i=1}^{N} j^*(i)\, \mathrm{A}^{(+)}(i) \prod_{i'=N+1}^{2N} j(i')\, \mathrm{A}^{(-)}(i') | 0\rangle. \tag{25.21}$$

Shifting all the operators $\mathrm{A}^{(-)}$ to the left and all the operators $\mathrm{A}^{(+)}$ to the right and using the property of the vacuum state, namely,

$$\mathrm{A}^{(+)} | 0\rangle = 0, \quad \langle 0 | \mathrm{A}^{(-)} = 0,$$

we arrive at the following expression for the commutation function (see Section 20):

$$[\mathrm{A}_{\mu}^{(+)}(x), \mathrm{A}_{\nu}^{(-)}(x')] = -4\pi g_{\mu\nu} D_+(x - x').$$

There are $N!$ such terms. As a result we find that

$$P(N) = \frac{e^{-\nu_1}}{(N!)^2} N! \left(-4\pi \int dx_1 \int dx_2 j^*(1)\, D_+(12)\, j(2)\right)^N. \tag{25.22}$$

Recalling formula (20.15) for the commutation function D_+, we can evaluate the integrals in (25.22):

$$-4\pi \int dx_1 \int dx_2 j^*(x_1)\, D_+(x_1 - x_2)\, j(x_2) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{\omega} | j^{\mu}(k) |^2, \tag{25.23}$$

where

$$| j^{\mu}(k) |^2 = | j^1 |^2 + | j^2 |^2 + | j^3 |^2 - | j^0 |^2, \tag{25.24}$$

and

$$j_\mu(k) = \int d^4x e^{ikx} j_\mu(x) \tag{25.25}$$

is the Fourier transform of the 4-current density.

Let us find the real part of (25.11),

$$\nu_1 = \operatorname{Re} \nu = \operatorname{Re}\left(-i \int dx_1\, dx_2 j(x_1) D(x_1 - x_2) j(x_2)\right). \tag{25.26}$$

Using the integral representation (25.4) of the D-function and the fact that

$$\frac{1}{k^2 + i\varepsilon} = \mathcal{P}\frac{1}{k^2} - i\pi\delta(k^2)$$

(see (13.28)), we get

$$\nu_1 = \frac{1}{(2\pi)^3} \int d^4k \delta(k^2) |\, j(k)\,|^2. \tag{25.27}$$

Integration with respect to k_0 in the last integral is carried out over the two halves of the light cone $k^2 = 0$, that is, $k^0 > 0$ and $k^0 < 0$, which yields

$$\int d^4k \delta(k^2) = \int \frac{d^3k}{\omega} \quad (\omega = |\,\mathbf{k}\,|), \tag{25.28}$$

whereby

$$\nu_1 = \frac{1}{(2\pi)^2} \int \frac{d^3k}{\omega} |\, j(k)\,|^2, \tag{25.29}$$

and this coincides with (25.23).

Thus, for the probability we have

$$P(N) = \frac{e^{-\nu_1}}{N!} \nu_1^N. \tag{25.30}$$

We see that the probability of a classical current emitting N photons is determined by the well-known Poisson distribution for random processes. The Poisson distribution is valid for statistically independent discrete processes, which means that the emission of each separate photon does not depend on whether other photons were emitted or not. This is possible, obviously, only when the photons have no effect on the source (no recoil), which is not surprising since the source is considered classical. In a real problem with a quantum source, an electron-positron current, such a model may be used to describe the emission of soft photons, whose effect on the source can be ignored.

The Poisson distribution (25.30) is normalized to unity,

$$\sum_N P(N) = 1, \tag{25.31}$$

while the average number of photons proves to be

$$\bar{N}=\sum_{N=1}^{\infty} NP(N)=\sum_{N=1}^{\infty} Ne^{-\nu_1}\frac{\nu_1^N}{N!}=\nu_1, \tag{25.32}$$

that is,

$$\bar{N}=\frac{1}{(2\pi)^2}\int\frac{d^3k}{\omega}\,|j(k)|^2. \tag{25.33}$$

Using (20.4), we can rewrite this formula as follows:

$$\bar{N}=\frac{2\pi}{L^3}\sum_{\mathbf{k}}\frac{1}{\omega}\,|j(k)|^2, \tag{25.34}$$

from which we can find the average number of photons with momentum $\mathbf{k}$ per unit volume,

$$\bar{N}_{\mathbf{k}}=\frac{1}{L^3}\frac{2\pi}{\omega}\,|j(k)|^2. \tag{25.35}$$

The average number of photons with $\mathbf{k}$ in the interval d^3k is

$$d\bar{N}=\frac{1}{(2\pi)^2}\frac{d^3k}{\omega}\,|j(k)|^2. \tag{25.36}$$

The angular spectral distribution of the radiation energy can be found by multiplying the number of photons by the photon energy ω:

$$d\varepsilon(\mathbf{n},\omega)=\frac{d^3k}{(2\pi)^2}\,|j(k)|^2, \tag{25.37}$$

with $\mathbf{n}=\mathbf{k}/\omega$.

This formula coincides with the classical quantity (14.60), which we calculated earlier using classical electrodynamics methods. The solution of the problem of emission of photons by a classical current has been obtained without resorting to perturbation methods, that is, the solution is exact in the interaction. This manifests itself in the fact that (25.30) is written in the form of an exponential, $\exp(-\nu_1)$, which can be expanded in a series for all $\nu_1=\bar{N}<\infty$, since an exponential function has an infinite convergence radius. Perturbation theory holds only for $\bar{N}<1$, that is, only when the average number of emitted photons is less than unity and processes in which one, two, or several photons are emitted can be separated, with one-photon emission being the most probable.

Let us consider an example in which the above results will be used. Suppose that a charged particle moves according to the following law specified in invariant form:

$$x_\mu(\tau)=\tau u_\mu\theta(-\tau)+\tau u'_\mu\theta(\tau), \tag{25.38}$$

where τ is the proper time, and u and u' are values of the 4-velocity at $\tau < 0$ and $\tau > 0$, respectively. The velocity changes abruptly at $\tau = 0$. In such an approximate manner we can describe, say, the scattering of electrons on a nucleus, since the time it takes for soft photons to form is much longer than the characteristic scattering time.

The Fourier transform of the current density, $j_\mu(k)$, that corresponds to the law of motion (25.38) is calculated via (16.44). We get

$$j_\mu(k) = eu \int_{-\infty}^{0} d\tau e^{iku\tau} + eu' \int_{0}^{\infty} d\tau e^{iku'\tau}. \tag{25.39}$$

Using formulas similar to (13.25), that is,

$$\begin{aligned} \int_0^\infty e^{i\tau(x+i\varepsilon)}\, d\tau &= i\mathcal{P}\frac{1}{x} + \pi\delta(x), \\ \int_{-\infty}^0 e^{i\tau(x-i\varepsilon)}\, d\tau &= -i\mathcal{P}\frac{1}{x} + \pi\delta(x), \end{aligned} \tag{25.40}$$

and bearing in mind that

$$\delta(uk) = \delta(u'k) = 0$$

at $k^2 = 0$ and $u^2 = u'^2 = 1$, we get

$$j_\mu(k) = ie\left(\frac{p'_\mu}{p'k} - \frac{p_\mu}{pk}\right), \tag{25.41}$$

where $p^\mu = mu^\mu$ and $p'^\mu = mu'^\mu$ are the 4-momenta before and after the scattering process. Note that $j_\mu(k)$ given by (25.41) obeys the continuity condition

$$kj(k) = 0. \tag{25.42}$$

Substituting (25.41) into (25.33), we get the average number of photons

$$\overline{N} = -\frac{e^2}{(2\pi)^2} \int_{\omega_{\min}}^{\omega_{\max}} \frac{d^3k}{\omega} \left(\frac{p}{pk} - \frac{p'}{p'k}\right)^2, \tag{25.43}$$

where $\overline{N}$ is positive since $j(k)$ is a spacelike vector.

What are the limits of integration in (25.43)? The upper limit, obviously, is restricted by the initial energy of the particle, $\varepsilon = (\mathbf{p}^2 + m^2)^{1/2}$, precisely, $\omega_{\max}$ must be less than ε. As for the low-

er limit ω_{min}, by sending it to zero we arrive at a divergent integral for $\bar{N}$:

$$\bar{N} \propto \int_{\omega_{min}} \frac{d\omega}{\omega} \to \infty. \tag{25.44}$$

At the same time, as $\omega_{min} \to 0$, the probability of a finite number of photons, $N < \infty$, being emitted tends to zero:

$$P(N) = \frac{e^{-\bar{N}}(\bar{N})^N}{N!} \to 0.$$

Therefore, there is a nonzero probability that an infinite number of photons are emitted, which agrees with the normalization condition for the probability, (25.31). It can be demonstrated that ω_{min} can always be restricted by a small, but nonzero, quantity. Let us take the nonrelativistic case of $v \ll 1$ and transfer to the reference frame in which the scattered particle is at rest, $\mathbf{v}' = 0$. Then

$$\left(\frac{p}{pk} - \frac{p'}{p'k}\right)^2 \simeq \frac{(\mathbf{n}\cdot\mathbf{v})^2 - v^2}{\omega^2} = -\frac{v^2}{\omega^2}\sin^2\theta,$$

where θ is the angle between the direction of propagation of radiation, $\mathbf{n}$, and velocity $\mathbf{v}$. Substituting into (25.43), we get

$$\bar{N} = \frac{\alpha}{(2\pi)^2} v^2 \int_0^{\pi} d\theta \sin^3\theta \int_0^{2\pi} d\varphi \int_{\omega_{min}}^{\omega_{max}} \frac{d\omega}{\omega}$$

$$= \frac{2}{3}\frac{\alpha v^2}{\pi} \ln\frac{\omega_{max}}{\omega_{min}} = \frac{2}{3}\frac{\alpha v^2}{\pi} \ln\frac{m}{\omega_{min}}, \tag{25.45}$$

where we have put $\omega_{max} \sim m$ and $\alpha = e^2 = 1/137$. Equation (25.45) enables estimating the conditions of applicability of perturbation-theory techniques in the given problem. Putting $\bar{N} < 1$, we find that

$$\bar{N} \propto \alpha \ln\frac{m}{\omega_{min}} < 1. \tag{25.46}$$

Sending ω_{min} to zero, we find that $\bar{N} \to \infty$. We can now estimate the values of ω_{min} at which the magnitude of $\bar{N}$ is roughly unity:

$$\omega_{min} \leqslant me^{-137}$$

(e here is the base of natural logarithms), or for the wavelength,

$$\lambda_{max} = \frac{2\pi}{\omega_{min}} \geqslant e^{137}(1/m). \tag{25.47}$$

Since $1/m$ is the Compton wavelength (i.e. $\hbar/mc$), the last inequality yields the estimate $\lambda_{max} > 10^{44}$ cm, which is much greater than the radius of the Universe. Thus, actually we can always assume ω_{min}

to be a nonzero quantity and $\overline{N}$ not to be too large. Emission of many soft photons accompanies every process that involves charged particles. Since emission of a soft photon is not followed by recoil, it does not influence the main process. In this case the main process and the emission process are statistically independent, and the differential probability of the total process taking place (per unit time) with emission of N photons is determined by the product

$$dw_N = dw_0 P(N), \tag{25.48}$$

where dw_0 is the probability of the main process without photon emission. When the frequency ω is low, the resolving power of the instrumentation proves to be insufficient for recording the soft photons, with the result that the probability must be summed over such photons:

$$dw = dw_0 \sum_N P(N) \tag{25.49}$$

Summing over a large number of photons, we obtain

$$\sum_N P(N) = 1, \quad dw = dw_0, \tag{25.50}$$

that is, the probability of a process that allows for the emission of all the soft photons coincides with the probability calculated without allowing for photon emission.

26. Feynman Diagrams

In the general case, in contrast to the model problem discussed in the preceding section, the series for the S-matrix cannot be summed exactly. We are therefore forced to employ perturbation-theory techniques, considering this series to be asymptotic and ignoring all terms in the series except a few, which correspond to the chosen order of the perturbation theory. The structure of the S-matrix expansion is such that to analyze it, it has proved convenient to use the graphical method developed by Richard Feynman, who established a number of rules for assigning definite graphs, known as Feynman diagrams, to the different terms in the S-matrix expansion.

26.1 The Structure of the S-matrix Expansion

Let us discuss the properties of the transition matrix element in the nth-order perturbation theory,

$$\langle f|S_n|i\rangle = \frac{(-ie)^n}{n!}\langle f|\int dx_1 \ldots dx_n \times T\,[N\bar{\psi}_1\gamma A_1\psi_1)(N\bar{\psi}_2\gamma A_2\psi_2)\ldots(N\bar{\psi}_n\gamma A_n\psi_n)]|i\rangle. \tag{26.1}$$

Here the initial and final states, $|i\rangle$ and $|f\rangle$, are obtained through applying a certain number of fermion operators a_s and b_s and boson (electromagnetic) operators c_s to the vacuum state:

$$|i\rangle = \prod_i c_i^\dagger a_i^\dagger b_i^\dagger |0\rangle, \quad \langle f| = \langle 0| \prod_f b_f a_f c_f, \tag{26.2}$$

where the number of creation and annihilation operators corresponds to the number of particles with given values of the quantum numbers, the particles being in the initial and final states.

To transform the matrix elements still further, we must employ Wick's theorem. The S-matrix expansion is conveniently represented by Feynman diagrams. In such diagrams, each vertex corresponds to a variable of integration x; a ray ending at a vertex corresponds to operator $\psi(x)$; a ray going out of a vertex corresponds to operator $\bar{\psi}(x)$; and, finally, a dashed ray (without direction being specified) starting at a vertex corresponds to operator $A_\mu(x)$ (Figure 26.1). Thus, electron-positron lines are solid while photon lines are dashed. An nth-order diagram has n vertices.

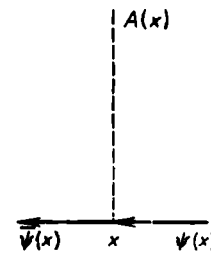

Figure 26.1

Expanding the S-matrix via Wick's theorem, we arrive at separate terms which contain several normally ordered creation and annihilation operators and several pairs of coupled operators, chronological pairings, which are expressed in terms of propagators. On a diagram the rays going out of and ending at a vertex correspond to creation and annihilation operators, while propagators correspond to the lines linking vertices. The lines corresponding to electron-positron propagators are directional, and their direction continues that of the fermion lines entering or leaving the vertices.

Let us take for an example the second-order perturbation theory, $n = 2$. Then, by Wick's theorem, for the second term in the S-matrix expansion, S_2, we get a sum of eight addends. One of these addends has an integrand without pairings, or

$$N(\bar{\psi}_1 \gamma A_1 \psi_1)(\bar{\psi}_2 \gamma A_2 \psi_2), \tag{26.3}$$

the second contains one pairing, $\overbrace{A_1 A_2}$, the third contains $\overbrace{\bar{\psi}_1 \psi_2}$, the fourth contains $\overbrace{\psi_1 \bar{\psi}_2}$, the fifth contains two pairings, $\overbrace{\bar{\psi}_1 \psi_2}$ and $\overbrace{A_1 A_2}$, the sixth contains $\overbrace{A_1 A_2}$ and $\overbrace{\psi_1 \bar{\psi}_2}$, the seventh contains $\overbrace{\bar{\psi}_1 \psi_2}$ and $\overbrace{\psi_1 \bar{\psi}_2}$, and the eighth contains three pairings, $\overbrace{\bar{\psi}_1 \psi_2}$, $\overbrace{\psi_1 \bar{\psi}_2}$, and $\overbrace{A_1 A_2}$. There also remains an appropriate number of free, uncoupled,

operators. The last addend does not contain free operators. The Feynman diagrams show the structure of S_2 most vividly (see Figure 26.2, where the diagrams correspond to the addends in the same order). The reader can see that diagrams *3* and *4* differ only in the permutation of their vertices. The corresponding addends are not distinct,

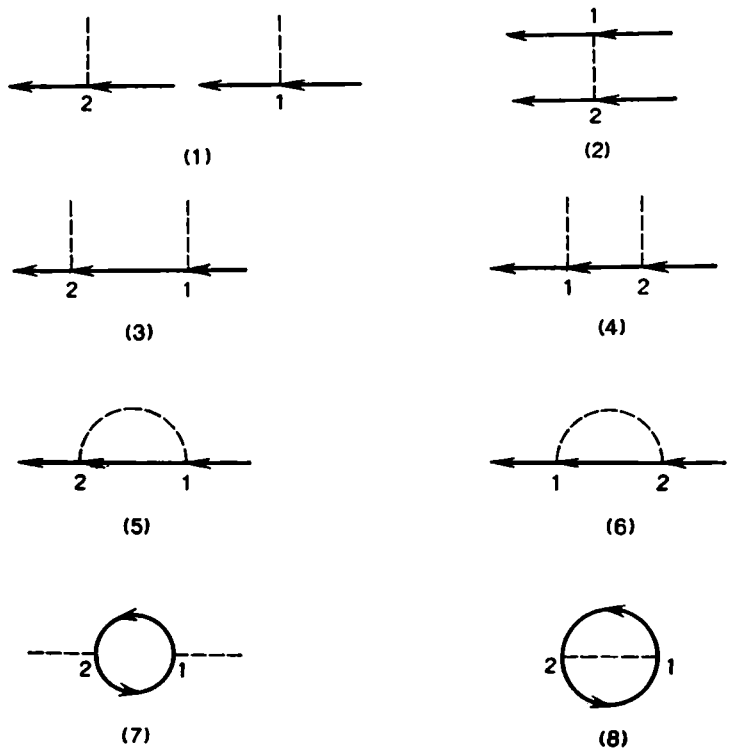

Figure 26.2

since one is obtained from the other by interchanging the variables of integration: $x_1 \to x_2$ and $x_2 \to x_1$. The same is true for diagrams *5* and *6*. For this reason we can always consider only one diagram in each pair and drop the common factor $1/2! = 1/2$ in S_2.

The higher-order terms in the S-matrix, $n > 2$, have a more complicated structure, but can always be made simple by using the Feynman diagrammatic technique.

26.2 Feynman's Rules

To calculate the matrix elements characterizing the transition of a system of particles from one state to another, we must apply plane-wave expansions to the Dirac and photon operators (see Sections 19

and 21). The operators A, ψ, and $\bar{\psi}$, normally ordered in accord with Wick's theorem, must be interchanged with the operators a, a^+, b, b^+, c, and c^+ present in the definition (26.2) of the initial and final states. After this permutation the annihilation operators $A^{(+)}$, $\psi^{(+)}$, and $\overline{\psi^{(-)}}$ will act directly on the vacuum state on the right and the corresponding negative-frequency operators, on the vacuum state on the left, giving zero, while the remaining commutators and anticommutators will be

$$[A_\mu^{(+)}(x),\ c_{k\lambda}^+] = \sqrt{\frac{2\pi}{\omega V}}\, e_\mu^{(\lambda)} e^{-ikx} \quad (\lambda = 1,\ 2), \tag{26.4a}$$

$$\{\psi^{(+)}(x),\ a_{p\sigma}^+\} = \frac{1}{\sqrt{2\varepsilon_p V}}\, u(p,\ \sigma)\, e^{-ipx}, \tag{26.4b}$$

$$\{\overline{\psi^{(-)}}(x),\ b_{p\sigma}^+\} = \frac{1}{\sqrt{2\varepsilon_p V}}\, \bar{u}(-p,\ \sigma)\, e^{-ipx} \tag{26.4c}$$

for positive-frequency operators and

$$[c_{k\lambda},\ A_\mu^{(-)}(x)] = \sqrt{\frac{2\pi}{\omega V}}\, e_\mu^{*(\lambda)} e^{ikx} \quad (\lambda = 1,\ 2), \tag{26.5a}$$

$$\{a_{p\sigma},\ \overline{\psi^{(+)}}(x)\} = \frac{1}{\sqrt{2\varepsilon_p V}}\, \bar{u}(p,\ \sigma)\, e^{ipx}, \tag{26.5b}$$

$$\{b_{p\sigma},\ \psi^{(-)}(x)\} = \frac{1}{\sqrt{2\varepsilon_p V}}\, u(-p,\ \sigma)\, e^{ipx}, \tag{26.5c}$$

for the negative-frequency operators.

Thus, such permutations select out of the field operators the components that correspond to the values of the quantum numbers of the initial and final particles. The propagators can also be expanded in four-dimensional Fourier integrals (see Sections 20 and 22). After integration over the coordinates at each vertex there appear delta functions, and the three momentum variables of the operators or propagators that meet at a vertex find themselves linked through a conservation law,

$$\int d^4x e^{i(p_1+p_2+k)x} = (2\pi)^4 \delta(p_1 + p_2 + k).$$

Since the fermion lines are solid, we find that after the "internal" momenta, corresponding to propagators, have been excluded, the sum of momenta for all initial particles is equal to the sum of momenta for all final particles, which is simply the 4-momentum conservation law

$$\sum_i p_i = \sum_f p_f. \tag{26.6}$$

Let us write the elements of the S-matrix as follows:

$$\langle f|S|i\rangle = \delta_{fi} + i(2\pi)^4 \delta\left(\sum_i p_i - \sum_f p_f\right) T_{fi}. \qquad (26.7)$$

The first term, δ_{fi}, corresponds to the absence of an interaction within the system of particles, that is, to the case where the initial and final states coincide. In the second term we have isolated the delta function responsible for 4-momentum conservation in the course of interaction. In what follows T_{fi} will always be called the amplitude of the process.

Thus, the matrix elements, or amplitudes, we arrive at have a structure that follows directly from the structure of the S-matrix. For this reason it is convenient to express the amplitudes in the form of Feynman diagrams, just as we did for the separate terms of the S-matrix expansion. But these new Feynman diagrams differ from the old in that now each external line (ray) is characterized by a definite set of quantum numbers of the particle—momentum and polarization, while the internal lines may have a definite momentum whose value follows from momentum conservation at a vertex or, if more than one internal line meet at a vertex, may not have such a value. Then we must evaluate the integral over the 4-momentum, which integral was left over from the integral representation of the propagator.

The correspondence between diagrams and amplitudes is established by rules that were formulated by Feynman.

(1) Each diagram corresponding to an amplitude has so many external incoming and outgoing lines (rays) as there are initial and final particles. The number of vertices in a diagram corresponds to the order of the perturbation theory (the number of the term in the S-matrix expansion).

(2) To each vertex there corresponds a matrix γ^μ.

(3) In accord with (26.4b) and (26.5c), to each incoming solid line there corresponds a positive-frequency bispinor

$$\frac{1}{\sqrt{2\varepsilon_p V}} u(p, \sigma),$$

which describes an electron in the initial state, or a negative-frequency bispinor

$$\frac{1}{\sqrt{2\varepsilon_p V}} u(-p, \sigma),$$

which describes a positron in the initial state. To each outgoing solid line, in accord with (26.4c) and (26.5b), there corresponds either a conjugate negative-frequency bispinor

$$\frac{1}{\sqrt{2\varepsilon_p V}} \bar{u}(-p, \sigma)$$

for a final positron or a conjugate positive-frequency bispinor

$$\frac{1}{\sqrt{2\varepsilon_p V}}\bar{u}(p, \sigma)$$

for a final electron.

(4) To each incoming dashed line there corresponds, in accord with (26.4a), a polarization 4-vector

$$\frac{\sqrt{4\pi}}{\sqrt{2\omega V}}e_\mu^{(\lambda)},$$

while to each outgoing dashed line there corresponds, in accord with (26.5a), a polarization 4-vector

$$\frac{\sqrt{4\pi}}{\sqrt{2\omega V}}e_\mu^{*(\lambda)},$$

with these vectors contracting with the appropriate γ-matrices at the vertices: $e_\mu\gamma^\mu$ and $e_\mu^*\gamma^\mu$. Note that in this case the dashed lines acquire direction.

(5) To each internal solid line there corresponds the Fourier transform of the spinor-field propagator,

$$\frac{i(\gamma p+m)}{p^2-m^2+i\varepsilon}=S_c(p), \tag{26.8}$$

while to each internal dashed line there corresponds the Fourier transform of the electromagnetic-field propagator taken with the opposite sign,

$$-\frac{4\pi i}{k^2+i\varepsilon}g_{\mu\nu}=-D^c_{\mu\nu}(k). \tag{26.9}$$

The photon propagator is contracted with the γ-matrices at the vertices that it connects: $\gamma^\mu D^c_{\mu\nu}\gamma^\nu$. The electron propagator is contracted in the bispinor indices with the γ-matrices at the vertices. Finally, a closed electron loop corresponds to the trace of the product of the bispinor matrices along this loop.

(6) The value of the 4-momentum of each internal line is determined from the momentum conservation law at the vertex. If more than one internal line gather at a vertex, the momentum may have no definite value, and then we must evaluate the integral with respect to this momentum with a factor $1/(2\pi)^4$. Note that the internal lines correspond to the so-called virtual particles, which are described by propagators instead of wave functions. Generally speaking, the momenta of such particles do not lie on the mass surface, that is, $p^2 \neq m^2$ for electron-positron lines and $k^2 \neq 0$ for photon lines. On the contrary, the momenta of the incoming and outgoing lines are fixed in such a manner that they satisfy the conservation law (26.6).

In accord with Feynman's rule (3), the positron momentum is taken equal to the argument of the bispinor with the opposite sign: $p_{\text{pos}} = -p$.

(7) To an nth-order amplitude we assign a factor $(-ie)^n$. An additional factor equal to -1 appears if the diagram has a closed fermion loop. The example in diagram 7 in Figure 26.2 clearly demonstrates the origin of this factor. As we said earlier, one of the pairings for this diagram is taken with the order in which the operators must follow reversed, $\overline{\psi_1 \psi_2}$. Returning to the correct order, with which the electron propagator was defined, $\overline{\psi_1 \psi_2} = -\overline{\psi_2 \psi_1}$, we arrive at the factor -1. The same factor appears in diagrams with two external positron ends connected by a single solid line. It originates in the rearrangement in normal order of the positron parts of operators $\bar{\psi}(x)$ and $\psi(x)$ in the S-matrix.

One amplitude may have corresponding to it several different diagrams of the same order and with the same number of external lines but with interchanged vertices of lines between them. All these diagrams must be added to obtain the total amplitude of the physical process. And only the total amplitude is gauge invariant. It can easily be verified that a correctly constructed amplitude satisfies the condition of gauge invariance. For this we must make sure that the substitution of $e_\mu + fk_\mu$ for e_μ, where k_μ is the 4-momentum of the photon, does not change the value of the amplitude.

If there are several identical fermions participating in the process, the diagrams with interchanged fermion lines enter the amplitude with opposite signs. Some concrete examples of application of the above-formulated rules will be given in the next chapter in calculating the simplest processes of quantum electrodynamics.

27. The Invariant Reaction Cross Section

In this section we will show how, using the reaction amplitudes, one can determine the physically measurable quantities, probabilities and cross sections.

27.1 Probability and Cross Section

The probability that a system that initially was in state $|\, i\rangle$ will be found in state $|\, f\rangle$ is equal, as we know, to the square of the modulus of the transition matrix element, $|\, \langle f \,|\, S \,|\, i\rangle|^2$. If we ignore the transitions with no change in state, then the first term, the unit matrix, in the expression (26.7) for the S-matrix can be dropped.

The square of the delta function can be written, if we employ the representation of the delta function

$$\delta(p)=\frac{1}{(2\pi)^4}\int d^4x e^{ipx}, \tag{27.1}$$

in the form

$$[\delta(p)]^2=\delta(p)\,\delta(0)=\delta(p)\,\frac{VT}{(2\pi)^4}, \tag{27.2}$$

where V is the three-dimensional volume and T is the time interval over which integration in (27.1) is carried out; both V and T are taken sufficiently large but finite. Note that the observable probabilities and cross sections must be independent of V and T.

The probability divided by the time interval T is the transition rate (probability of transition per unit time)

$$w_{fi}=\frac{|\langle f|S|i\rangle|^2}{T}. \tag{27.3}$$

According to what has been said, for the transition rate we have the following formula:

$$w_{fi}=(2\pi)^4\delta\left(\sum_i p_i-\sum_f p_f\right)|T_{fi}|^2V, \tag{27.4}$$

where T_{fi} is the amplitude of the process.

The reader will recall that, according to Feynman's rules, each initial or final particle contributes a factor $(2\varepsilon V)^{-1/2}$ to the amplitude, where ε is the particle energy ($\varepsilon=\omega$ for photons). Since these factors contain the normalization volume V, which has no direct physical meaning, it is advisable to isolate these factors in the amplitude and introduce a new amplitude A_{fi} as follows:

$$T_{fi}=\frac{A_{fi}}{\prod\limits_i\sqrt{2\varepsilon_iV}\prod\limits_f\sqrt{2\varepsilon_fV}}, \tag{27.5}$$

where the products in the denominator are over all initial and final quantum numbers.

In accord with this definition, there is no need to include the factors $(2\varepsilon V)^{1/2}$ in the A_{fi} when we construct these new amplitudes via Feynman's rules, whereas for T_{fi} we must do this.

Summation over the final states in (27.4) is carried out as follows. As is known, the states of free particles are quasi-classical. The number of quasi-classical states of a particle per momentum interval d^3p in volume V is $Vd^3p/(2\pi)^3$. For this reason, the probability of

a system of particles transforming into a finite state with momenta within the interval $\prod_f d^3p_f$ in volume V is

$$dw = (2\pi)^4\delta\left(\sum_i p_i - \sum_f p_f\right)|T_{fi}|^2 V \prod_f \frac{V\,d^3p_f}{(2\pi)^3}, \qquad (27.6)$$

or, in terms of the new amplitude A_{fi},

$$dw = (2\pi)^4\delta\left(\sum_i p_i - \sum_f p_f\right)\frac{|A_{fi}|^2 V}{\prod_i 2\varepsilon_i V}\prod_f \frac{d^3p_f}{2\varepsilon_f(2\pi)^3}. \qquad (27.7)$$

Here are two characteristic reactions. The first is the decay of a particle into several other particles. In this reaction there is one particle in the initial state, and the probability (27.7) assumes the form

$$dw = (2\pi)^4\delta\left(p_i - \sum_f p_f\right)\frac{|A_{fi}|^2}{2\varepsilon_i}\prod_f \frac{d^3p_f}{2\varepsilon_f(2\pi)^3}. \qquad (27.8)$$

The reader can see that the decay probability does not depend on the (nonphysical) normalization volume.

The second reaction has to do with the collision of two particles with energies ε_1 and ε_2 and momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. In this case it has proved expedient to introduce the notion of cross section. The differential cross section of a collision is defined as

$$d\sigma = j^{-1}dw, \qquad (27.9)$$

where j is the flux of particles. The definition (27.9) refers to a special reference frame, namely the reference frame connected with the center of inertia (or center of mass) of the system of the colliding particles ($\mathbf{p}_1 = -\mathbf{p}_2$). The flux is then

$$j = \frac{v_1 + v_2}{V}, \qquad (27.10)$$

where $v_1 = v_2$ are the velocities of the particles in the given frame.

A covariant generalization of the definition of flux can be given in the form

$$j = \frac{I}{V\varepsilon_1\varepsilon_2}, \qquad (27.11)$$

where

$$I = \sqrt{(p_1p_2)^2 - m_1^2m_2^2} \qquad (27.12)$$

is a scalar quantity. It can easily be verified that the definitions (27.11) and (27.10) coincide in the center-of-mass frame. The defi-

nition (27.9) then yields the following formula for the cross section:

$$d\sigma = (2\pi)^4 \delta\left(p_1 + p_2 - \sum_f p_f\right) \frac{|A_{fi}|^2}{4I} \prod_f \frac{d^3p_f}{(2\pi)^3 2\varepsilon_f}. \quad (27.13)$$

The expression obtained defines the invariant cross section of a reaction and does not depend on the normalization volume. The invariance of this expression is obvious, since the amplitude A_{fi} and the scalar quantity I are invariant. So are the volume elements in the momentum space and the four-dimensional delta function

$$\delta\left(p_1 + p_2 - \sum_f p_f\right) \prod_f \frac{d^3p_f}{2\varepsilon_f} = \text{inv},$$

since the three-dimensional integral $\int (2\varepsilon)^{-1} d^3p$ can be transformed into the four-dimensional integral $\int d^4p\delta\,(p^2 - m^2)$ taken over the upper surface of the light cone, $p^0 > 0$. The four-dimensional delta function is also Lorentz-invariant: $\delta\,(0) = VT = \text{inv}$.

Up till now we have dealt only with summation over spatial degrees of freedom of particles, the momenta, but we must also carry out summation over the spin states of the particles, the polarizations.

If the initial state is nonpolarized, we must also average the result over the initial spin states.

To carry out these operations, let us write the amplitude of the process as

$$A_{fi} = \bar{u}_f A u_i, \quad (27.14)$$

where u_i is the bispinor of the initial state and u_f is the bispinor of the final state. The square of the modulus is

$$|A_{fi}|^2 = u_i^\dagger A^+ \bar{u}_f^\dagger \bar{u}_f A u_i = \bar{u}_i \bar{A} u_f \bar{u}_f A u_i, \quad (27.15)$$

where $\bar{A} = \gamma^0 A^+ \gamma^0$. If we write out all the bispinor indices in (27.15), we get

$$|A_{fi}|^2 = (\bar{u}_i)_{\alpha\beta} (\bar{A})_{\beta\gamma} (u_f)_{\gamma\mu} (\bar{u}_f)_{\mu\nu} A_{\nu\rho} (u_i)_{\rho\alpha},$$

which is the trace of a matrix product:

$$|A_{fi}|^2 = \mathrm{Tr}\,(\bar{u}_i A u_f \bar{u}_f A u_i). \quad (27.16)$$

By virtue of a property of a trace, the cofactors in (27.16) allow for cyclic permutations:

$$|A_{fi}|^2 = \mathrm{Tr}\,(u_i \bar{u}_i \bar{A} u_f \bar{u}_f A). \quad (27.17)$$

Let us now recall the following identity for positive-frequency bispinors (see Section 21):

$$\sum_{\sigma} u(p, \sigma)\,\bar{u}(p, \sigma) = \gamma p + m. \tag{27.18}$$

Thus, if the initial and final states have one electron each, then summing over the final polarizations and averaging over the initial polarizations yield

$$\frac{1}{2}\sum_{\sigma_i,\,\sigma_f} |A_{fi}|^2 = \frac{1}{2}\,\mathrm{Tr}\,\{(\gamma p_i + m)\,\bar{A}\,(\gamma p_f + m)\,A\}. \tag{27.19}$$

For negative-frequency bispinors the identity (27.18) has a "minus" sign in front of m. For this reason, if among the initial or final particles there is a positron, we must change the sign in front of m in the appropriate factor in (27.19). When there is more than one spinor particle in the initial and/or final states, summation over polarizations leads to expressions that are more complicated than (27.19), but the method of calculation remains the same.

As for polarization of photons, it is described by the 4-vectors e_μ. Each photon contributes a factor $e_\mu e_\nu^*$ to the square of the amplitude. The reader will recall that, in the case of three-dimensional transverse gauge, summation over polarizations is carried out via (16.64),

$$\sum_{\lambda=1,\,2} e_i^{(\lambda)} e_k^{*(\lambda)} = \delta_{ik} - n_i n_k, \tag{27.20}$$

where $i, k = 1, 2, 3$, $n_i = k_i/\omega$, and, obviously, $e_0^{(\lambda)} = 0$ (complex conjugation refers to the case of circular polarization, which is described by complex-valued vectors).

According to Feynman's rules, the polarization vectors are encountered in the amplitude in combination with the transition currents $j_{21}(k) = \bar{u}_2(p_2)\,\gamma u_1(p_1)$, that is, in the form

$$e j_{21}(k) = \bar{u}_2(p_2)\,e_\mu \gamma^\mu u_1(p_1), \tag{27.21}$$

where $k = p_1 - p_2$. For this reason, the sum (27.20) yields

$$\sum_{\lambda} (\mathbf{j}_{21}\cdot\mathbf{e}^{(\lambda)})\,(\mathbf{j}_{21}^*\cdot\mathbf{e}^{*(\lambda)}) = |\mathbf{j}_{21}|^2 - |\mathbf{j}_{21}\cdot\mathbf{n}|^2. \tag{27.22}$$

Expression (27.22) is gauge invariant, that is, does not change under transformations $e \to e' = e + fk$. This means that we must have

$$(j_{21}k) = 0, \tag{27.23}$$

which is simply the continuity condition for the transition current, $\partial_\mu j^\mu = 0$, written in the momentum representation. From (27.23) we arrive at

$$j_{21}^0 k^0 = \mathbf{j}_{21} \cdot \mathbf{k}, \tag{27.24}$$

whereby (27.22) assumes the form

$$\sum_{\lambda=1,\,2} (\mathbf{j}_{21} \cdot \mathbf{e}^{(\lambda)}) (\mathbf{j}_{21}^* \cdot \mathbf{e}^{*(\lambda)}) = -(j_{21})^\mu (j_{21}^*)_\mu = |j_{21}|^2. \tag{27.25}$$

In this expression we can go over to any other gauge of the polarization vectors in which the temporal coordinates may be nonzero, too. Then, in view of the gauge invariance we obtain

$$\sum_\lambda (j_{21} e^{(\lambda)}) (j_{21}^* e^{*(\lambda)}) = |j_{21}|^2, \tag{27.26}$$

that is,

$$\sum_\lambda e_\mu^{(\lambda)} e_\nu^{*(\lambda)} = -g_{\mu\nu}. \tag{27.27}$$

Averaging over two transverse polarizations differs from (27.20) in that we have a factor of 1/2, that is,

$$\overline{e_i e_k^*} = \frac{1}{2} \sum_{\lambda=1,\,2} e_i^{(\lambda)} e_k^{*(\lambda)} = \frac{1}{2} (\delta_{ik} - n_i n_k). \tag{27.28}$$

Transition to other gauges does not, obviously, change this factor, since temporal and longitudinal photons cannot be observed. Thus, for the general case we have

$$\overline{e_\mu e_\nu^*} = -\frac{1}{2} g_{\mu\nu}. \tag{27.29}$$

27.2 Optical Theorem

The transition matrix $\langle f \mid \mathrm{S} \mid i \rangle$ must satisfy the obvious normalization condition for the probability

$$\sum_f |\langle f|\mathrm{S}|i\rangle|^2 = 1, \tag{27.30}$$

or

$$\sum_f \langle i|\mathrm{S}|f\rangle \langle f|\mathrm{S}|i\rangle = 1. \tag{27.31}$$

This condition follows from the general condition of unitarity of the S-matrix operator

$$\mathrm{S}^+\mathrm{S} = \mathrm{I}, \tag{27.32}$$

which ensures the conservation of probability and the orthogonality of the states.

Let us write (27.32) in the form

$$\sum_f \langle i|S^+|f\rangle \langle f|S|i'\rangle = \delta_{ii'}. \tag{27.33}$$

Substituting the matrix elements (26.7), we obtain

$$T_{ii'} - T^*_{i'i} = i(2\pi)^4 \sum_f \delta\left(\sum_i p_i - \sum_f p_f\right) T^*_{fi} T_{fi'}. \tag{27.34}$$

This condition, obviously, relates the matrix elements in different orders of the perturbation theory. It can be employed to determine the probabilities or cross sections expressed in terms of quadratic combinations of amplitudes through linear combinations of higher-order amplitudes. Take, for example, the case where $i = i'$. This means that amplitude T_{ii} describes an elastic process in which the state of the system does not change. Then (27.34) yields

$$2 \operatorname{Im} T_{ii} = (2\pi)^4 \sum_f |T_{fi}|^2 \delta\left(\sum_i p_i - \sum_f p_f\right). \tag{27.35}$$

As the reader can see, on the right we have, to within a factor, the total probability of a system transferring from state $|i\rangle$ to all states $|f\rangle$, while on the left we have the imaginary part of the amplitude of the elastic process with no change in state, $|i\rangle \to |i\rangle$.

According to (27.4), we have

$$2 \operatorname{Im} T_{ii} = \frac{1}{V} \sum_f w_{fi}, \tag{27.36}$$

that is, Im T_{ii} is expressed in terms of the total transition probability per unit time (total transition rate). Going over to the cross section via (27.9), we obtain

$$\frac{2V}{j} \operatorname{Im} T_{ii} = \sigma_{\text{tot}}, \tag{27.37}$$

where σ_{tot} is the total cross section. On the left we have the imaginary part of the elastic scattering amplitude without change in state, or scattering by a zero angle. This result, which expresses the total probability or cross section in terms of the imaginary part of the diagonal matrix element, is known as the optical theorem.

Chapter IV

The Interaction of Electrons with Photons

Let us consider several concrete effects of quantum electrodynamics using, as a basis, the general theory discussed in the previous chapter. The simplest process, which emerges already in first-order perturbation theory in the electromagnetic interaction, is the emission of a photon by an electron, or electromagnetic radiation. One important application of quantum theory considered in this chapter is synchrotron radiation, whose classical theory was developed in Chapter II. After that we study second- and third-order effects: the scattering of a photon on an electron, creation and annihilation of electron-positron pairs, and bremsstrahlung of electrons at a nucleus. In the final section of this chapter we briefly consider the structure of higher-order terms in the perturbation theory expansion, the divergences that emerge, and the methods of excluding such divergences that involve the renormalization of electron mass and charge.

28. The Quantum Theory of Radiation

In classical electrodynamics (see Chapter II), a uniformly moving electron does not emit radiation, that is, emission requires that the electron be accelerated or decelerated (say by an external field).* From the viewpoint of quantum theory this means that a free electron cannot emit a photon. Indeed, according to the energy-momentum conservation law,

$$p = p' + k, \tag{28.1}$$

where p and p' are the 4-momenta of the electron in the initial and final states, and k is the photon momentum. Squaring (28.1) (i.e. forming a scalar product of both sides by the respective sides) and

* If an electron is moving in a medium instead of a vacuum, radiation may appear in uniform electron motion. This happens when the electron velocity is greater than the phase velocity of light in that medium (the Cherenkov effect).

allowing for the fact that $p^2 = p'^2 = m^2$ (where m is the electron mass) and $k^2 = 0$, we get $p'k = 0$, which is impossible since for $\omega = k_0 > 0$ we have $p'k = \varepsilon'\omega(1 - \mathbf{v}' \cdot \mathbf{n}) > 0$, by virtue of $|\mathbf{v}'| < 1$ and $|\mathbf{n}| = 1$ (here $\mathbf{v}' = \mathbf{p}'/\varepsilon'$ is the electron velocity and $\mathbf{n} = \mathbf{k}/\omega$ is the unit vector pointing in the direction of the photon momentum). We arrive at the same conclusion immediately if we consider the emission act in a reference frame in which the initial electron is at rest. The energy of this electron is minimal (equal to m) and cannot be made smaller. But if the electron moves in an external field, its momentum is not conserved and photon emission can occur.

28.1 The Furry Representation

Suppose that an electron interacts not only with a quantized electromagnetic field but also with an external field that is described in classical terms and is considered given.* The notion of a given external field is, of course, of limited value because in strong fields there can be spontaneous creation of electron-positron pairs (out of the vacuum), which is a process that lies outside the scope of the one-electron problem (see [7, 8]). Besides, even a weak electromagnetic field cannot be described in classical terms since it contains only a few photons and this number changes in the course of interaction with the electron.

The motion of an electron in a given external field is described by a Dirac equation that is a generalization of the appropriate equation for a free electron and has the form (see (10.16))

$$[\gamma_\mu (p^\mu - eA^\mu_{ext}) - m]\psi = 0, \tag{28.2}$$

where γ_μ are the Dirac matrices (see (8.48)), $p^\mu = i\partial^\mu = (i\partial/\partial t, -i\nabla)$ is the 4-momentum operator, and $A^\mu_{ext} = (\Phi_e, \mathbf{A}_e)$ is the 4-potential of the external field. Introducing, like we did in Section 8, the matrices

$$\alpha = \gamma^0\gamma, \quad \beta \equiv \rho_3 = \gamma^0, \tag{28.3}$$

we can write the Dirac equation in the Hamiltonian form (here we use common units)

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi,$$
$$H = c\alpha\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}_e\right) + e\Phi_e + \rho_3 mc^2. \tag{28.4}$$

* This means that electron motion does not affect the motion of the particles serving as the source of the external field.

Quantization of the Dirac field that obeys (28.2) is carried out in exactly the same manner as quantization of the free field. The operator field function ψ can be represented in the form of expansion

$$\begin{aligned} \psi(x) &= \sum_n [a_n \psi_n^{(+)}(x) + b_n^+ \psi_n^{(-)}(x)], \\ \bar{\psi} &= \sum_n [a_n^+ \bar{\psi}_n^{(+)}(x) + b_n \bar{\psi}_n^{(-)}(x)], \end{aligned} \tag{28.5}$$

where $\psi_n^{(\pm)}(x) = \psi_n^{(\pm)}(\mathbf{r}) \exp(\mp i\varepsilon_n^{(\pm)} t)$ are the normalized exact solutions of the Dirac equation for an electron in an external field, (28.2), solutions that refer to positive ($\varepsilon_n^{(+)}$) and negative ($-\varepsilon_n^{(-)}$) frequencies; n stands for the set of quantum numbers that fix the state of the electron (this set replaces the momentum $\mathbf{p}$ and the spin σ, which define the plane wave representing a free particle; see Section 22); $\bar{\psi} = \psi^+\gamma^0$ is the Dirac-conjugate function; and a_n^+ (b_n^+) and a_n (b_n) are the electron (positron) creation and annihilation operators (the electrons and positrons are in state n). Moreover, just like we did in Section 23, we introduce the interaction of the electron with the quantized radiation field A^μ, which can be taken into account through perturbation techniques. The appropriate Feynman rules are formulated in the same way as in Section 26, the only difference being that the wave functions of free particles (plane waves) must be replaced with the solutions $\psi_n^{(\pm)}$ of the Dirac equations in an external field, (28.2). The picture of the interaction of the electron with the electromagnetic field in which the external field (described in classical terms) is taken into account exactly while the quantized radiation field is allowed for by perturbation-theory techniques is known as the Furry representation.

28.2 Spontaneous Emission

In lower-order perturbation theory, the emission of a photon by an electron in a given external field is described as an electron transition from state ψ_i to state ψ_f accompanied by the emission of a photon with a 4-momentum $k^\mu = (\omega, \mathbf{k})$ and a polarization 4-vector e^μ. The amplitude of such a transition is given by the following element of the S-matrix (see Section 26):

$$S_{fi} = \langle 1_{\mathbf{k}}; f \mid S \mid 0_{\mathbf{k}}; i \rangle, \tag{28.6}$$

where it was assumed that in the initial state there are no photons of the given type (a spontaneous transition). According to Feynman's rules (Section 26), we can write

$$S_{fi} = -ie\sqrt{4\pi} \int d^4x \bar{\psi}_f(x) \gamma^\mu e_\mu^* \frac{e^{ikx}}{\sqrt{2\omega L^3}} \psi_i(x), \tag{28.7}$$

where L^3 is the normalization volume. The respective Feynman diagram is depicted in Figure 28.1, where the double electron lines show that the exact wave functions of an electron in an external field are employed (the Furry representation).

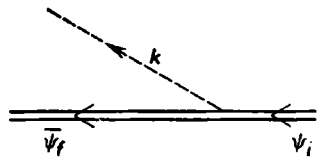

Figure 28.1

In what follows we assume that the external field is time independent. Then the wave functions of the stationary states of the electron are $\psi_n(x) = e^{-i\varepsilon_n t}\psi_n(\mathbf{r})$, and integration in (28.7) with respect to time yields

$$S_{fi} = -ie\sqrt{\frac{2\pi}{\omega L^3}}\, 2\pi\delta(\varepsilon_f + \omega - \varepsilon_i)(e^*_\mu j^\mu_{fi}(\mathbf{k})); \tag{28.8}$$

here

$$j^\mu_{fi}(\mathbf{k}) = \int d^3x \bar{\psi}_f(\mathbf{r})\gamma^\mu e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_i(\mathbf{r}), \tag{28.9}$$

and the delta function stands for energy conservation in a time independent external field.

The expression for the transition rate

$$w_{fi}(\mathbf{k}) = \lim_{T\to\infty} |S_{fi}|^2/T \tag{28.10}$$

follows from (28.8) combined with the fact that

$$[2\pi\delta(\varepsilon_f + \omega - \varepsilon_i)]^2 = 2\pi\delta(\varepsilon_f + \omega - \varepsilon_i)\lim_{T\to\infty} T$$

(see Section 16.3). The result is

$$w_{fi}(\mathbf{k}) = 2\pi\delta(\varepsilon_f + \omega - \varepsilon_i)\frac{2\pi e^2}{\omega L^3}|e^* j_{fi}(\mathbf{k})|^2. \tag{28.11}$$

The total transition rate is

$$w_{fi} = \sum_{\mathbf{k}} w_{fi}(\mathbf{k}) \tag{28.12}$$

(only the $i \to f$ transition is considered). We wish to transform the sum over $\mathbf{k}$ into an integral via the formula

$$\sum_{\mathbf{k}} F(\mathbf{k}) = L^3 \int \frac{d^3k}{(2\pi)^3} F(\mathbf{k}). \tag{28.13}$$

We then substitute (28.11) into (28.12), which yields

$$w_{fi} = \frac{e^2}{2\pi} \int \frac{d^3k}{\omega} \delta(\omega - \omega_{fi}) |e^* j_{fi}(\mathbf{k})|^2, \tag{28.14}$$

where the transition frequency is

$$\omega_{fi} = \varepsilon_i - \varepsilon_f. \tag{28.15}$$

For the photon wave function $A^\mu(x) = (2\omega L^3)^{-1/2} e^\mu \exp(-ikx)$ in (28.7) we select the three-dimensional transverse gauge. In this gauge, $e^\mu = (0, \mathbf{e})$, and the transition rate (28.14) can be expressed in terms of three-dimensional quantities thus:

$$w_{fi}^{(\lambda)} = \frac{e^2}{2\pi} \int \frac{d^3k}{\omega} \delta(\omega - \omega_{fi}) |\mathbf{e}_{(\lambda)}^* \cdot \mathbf{j}_{fi}|^2, \tag{28.14a}$$

$$\mathbf{j}_{fi} = \int \psi_f^*(\mathbf{r}) \alpha e^{-i\mathbf{k}\cdot\mathbf{r}} \psi_i(\mathbf{r}) \, d^3x,$$

where we have introduced the label $\lambda = 1, 2$ that numbers the two independent polarization states (see Section 16.4). The integrand in (28.14a) determines the probability of an electron emitting a photon with a given polarization $\mathbf{e}_{(\lambda)}$ in the direction $\mathbf{n} = \mathbf{k}/\omega$, with the photon frequency in the given transition $i \to f$ being fixed by the energy conservation law (28.15).

Let us now sum over the polarization states of the photon via (16.64), which can now be conveniently written in the form

$$\sum_\lambda (\mathbf{a}\cdot\mathbf{e}_{(\lambda)})(\mathbf{b}\cdot\mathbf{e}_{(\lambda)}) = (\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{n})(\mathbf{b}\cdot\mathbf{n}) \equiv (\mathbf{a}\times\mathbf{n})(\mathbf{b}\times\mathbf{n}), \tag{28.16}$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors.

Combining (28.16) with (28.14a), we can find the transition rate summed over the polarizations:

$$w_{fi} = \frac{e^2}{2\pi} \int \frac{d^3k}{\omega} \delta(\omega - \omega_{fi}) |\mathbf{j}_{fi} \times \mathbf{n}|^2. \tag{28.17}$$

Integrating with respect to ω via the delta function yields the angular distribution of the radiation:

$$\frac{dw_{fi}}{d\Omega} = \frac{e^2}{2\pi} \omega |\mathbf{j}_{fi} \times \mathbf{n}|^2, \tag{28.18}$$

where $d\Omega$ is the solid angle element in the direction of $\mathbf{n}$, and $\omega = \omega_{fi}$.

The radiated power at $\omega = \omega_{fi}$ can be obtained by multiplying the transition rate w_{fi} by $\hbar\omega_{fi}$ (since $\tau_{fi} = \hbar/\omega_{fi}$ has the meaning of the mean lifetime of the electron on level ε_i with respect to the $i \to f$ transition, that is, this quantity is the time interval during which the emission of a single photon with energy $\hbar\omega_{fi}$ occurs):

$$W(\omega_{fi}) = \omega_{fi} w_{fi}. \tag{28.19}$$

Substitution of (28.14a) yields a formula for the power carried by polarized radiation:

$$W^{(\lambda)}(\omega_{fi}) = \frac{e^2}{2\pi} \int d^3k\delta(\omega - \omega_{fi}) |\, \mathbf{e}^*_{(\lambda)} \cdot \mathbf{j}_{fi}|^2. \qquad (28.20)$$

The formula for the angular distribution of radiation follows from (28.18):

$$\frac{dW(\omega_{fi})}{d\Omega} = \frac{e^2}{2\pi} \omega_{fi}^3 |\mathbf{j}_{fi} \times \mathbf{n}|^2. \qquad (28.21)$$

The total radiated power can be obtained from (28.19) by summing over all final states of the electron:

$$W = \sum_f W(\omega_{fi}). \qquad (28.22)$$

Let us now compare the formulas of the quantum theory of radiation with those of the classical theory (see Section 16). According to the correspondence principle in the region of large quantum numbers, where the states ψ_i and ψ_f are quasi-classical, the results of the quantum theory must transfer into classical results. Indeed, quantum mechanics states that quasi-classical matrix elements are approximately the Fourier transforms of the corresponding classical quantities. For this case, in the quasi-classical approximation, we have (see (28.14a), with α the velocity operator)

$$\mathbf{j}_{fi} \simeq \frac{1}{T} \int_0^T \mathbf{v}(t) \exp[il_{fi}\omega_0 t - i\mathbf{k} \cdot \mathbf{r}(t)]\, dt, \qquad (28.23)$$

where $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are the classical position and velocity vectors of the electron, $l_{fi} \simeq (\varepsilon_i - \varepsilon_f)/\hbar\omega_0$ is an integer, $\hbar\omega_0$ is the energy gap between two quasi-classical neighboring levels, and $T = 2\pi/\omega_0$ is the period of the finite classical motion (corresponding to a stationary state in quantum mechanics) with energy $\varepsilon \simeq \varepsilon_f \simeq \varepsilon_i$ (in the quasi-classical approximation, $\hbar\omega_{fi} \ll \varepsilon_f, \varepsilon_i$). Substituting (28.23) into (28.20), we arrive at the classical formula (16.61) for the radiated power of the lth harmonic ($l \equiv l_{fi}$) of the fundamental frequency ω_0 of the motion.

The correspondence between the classical and quantum theories can also be established by a more pictorial method (from the standpoint of physics; see [8, 11]). Let us write the S-matrix element (28.7) using the three-dimensional gauge for the photon polarization vector in the form

$$S_{fi} = ie(2\pi/\omega L^3)^{1/2} \int_{-\infty}^{\infty} dt \langle f | e^{iHt} (\alpha \cdot \mathbf{e}^*) e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} e^{-iHt} | i \rangle, \quad (28.24)$$

where H is the Hamiltonian of an electron in an external field (see (28.4)), which by hypothesis is time independent (the field is time independent), and

$$|n\rangle = \psi_n(\mathbf{r}) \quad \text{and} \quad \psi_n(t, \mathbf{r}) = e^{-ie_n t}\psi_n(\mathbf{r}) = e^{-iHt}|n\rangle$$

($n = i, f$). Let us introduce the Heisenberg operators (which are time dependent) corresponding to the position vector $\mathbf{r}(t)$ and the velocity vector $\mathbf{v}(t)$ [3, Sec. 7]:

$$\mathbf{v}(t) = e^{iHt}\alpha e^{-iHt}, \quad e^{-i\mathbf{k}\cdot\mathbf{r}(t)} = e^{iHt}e^{-i\mathbf{k}\cdot\mathbf{r}}e^{-iHt}. \tag{28.25}$$

These operators transform the matrix element (28.24) thus:

$$S_{fi} = ie(2\pi/\omega L^3)^{1/2}\int_{-\infty}^{\infty} dt\,\langle f|R(t)|i\rangle,$$
$$R(t) = (\mathbf{v}(t)\cdot\mathbf{e}^*)\exp[i\omega t - i\mathbf{k}\cdot\mathbf{r}(t)]. \tag{28.26}$$

The total probability of an electron transferring from state $|i\rangle$ into all other allowed states and in the process emitting a photon with momentum $\mathbf{k}$ and polarization $\mathbf{e}$ is

$$\mathcal{P}(\mathbf{k}) = \sum_f |S_{fi}|^2 = \frac{2\pi e^2}{\omega L^3}\sum_f \int dt_1 \int dt_2\,\langle i|\mathrm{R}^+(t_2)\,\mathrm{R}(t_1)|i\rangle, \tag{28.27}$$

where in summing over the finite states $|f\rangle$ we employ the completeness condition $\sum_f |f\rangle\langle f| = \mathrm{I}$ (here I is the unit operator; in the coordinate representation the completeness condition implies $\sum_f \psi_f(\mathbf{r})\psi_f^*(\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')$). This leads to an expression for the angular spectral distribution of the radiation energy (see (28.13))

$$d\varepsilon = \omega\mathcal{P}(\mathbf{k})\frac{L^3 d^3k}{(2\pi)^3} \tag{28.28}$$

in the form

$$d\varepsilon = \frac{e^2}{(2\pi)^2}d^3k\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} dt_2 e^{i\omega(t_1 - t_2)}$$
$$\times\langle i|(\mathbf{v}(t_2)\cdot\mathbf{e})\exp[i\mathbf{k}\cdot\mathbf{r}(t_2)](\mathbf{v}(t_1)\cdot\mathbf{e}^*)\exp[-i\mathbf{k}\cdot\mathbf{r}(t_1)]\,|i\rangle. \tag{28.29}$$

This formula has, of course, only symbolic meaning since all the Heisenberg operators in it are noncommutative and the calculation of the average $\langle i|\ldots|i\rangle$ requires, in the final analysis, knowing the explicit form of the operators and their spectra. However, the classical limit of (28.29) is crystal clear: it is simply formula (16.60a), and to obtain it we must only replace the Heisenberg operators $\mathbf{r}(t)$ and $\mathbf{v}(t)$ in (28.29) with the respective classical quantities.

28.3 Stimulated Emission and Absorption of Radiation

We now know that in spontaneous emission there are no photons in the initial state of the electron-field system. But suppose we have $N_{\mathbf{k}\lambda}$ photons with momentum $\mathbf{k}$ and polarization $\mathbf{e}_{(\lambda)}$ in the initial state of the system. The matrix element representing the $i \to f$ transition accompanied by emission of a photon of the same type differs from the matrix element (28.7) for the transition with spontaneous emission (see Section 19) by a factor:

$$\langle N_{\mathbf{k}\lambda}+1;\ j|\mathrm{S}|N_{\mathbf{k}\lambda};\ i\rangle = \sqrt{N_{\mathbf{k}\lambda}+1}\, S_{fi}. \tag{28.30}$$

Correspondingly, the transition rate is

$$w_{fi}^{(\mathrm{emis})} = (N_{\mathbf{k}\lambda}+1)\, w_{fi}, \tag{28.31}$$

where w_{fi} is the transition rate with spontaneous emission, (28.14). Thus, the presence of photons that are identical with the emitted photon only increases the probability of emission, which is a direct consequence of the Bose statistics governing the behavior of a collection of photons. The first term in (28.31) gives the probability of stimulated emission,

$$w_{fi}^{(\mathrm{stim})} = N_{\mathbf{k}\lambda} w_{fi}, \tag{28.32}$$

which emission is caused exclusively by the presence of photons in the system. For applications, it has proved convenient to express the number of such photons, $N_{\mathbf{k}\lambda}$, in terms of the energy flux of the radiation incident on the system, $I_{\mathbf{k}\lambda}$, with $I_{\mathbf{k}\lambda} d\omega d\Omega$ the energy of the radiation impinging every second on a surface of unit area in the frequency interval from ω to $\omega + d\omega$ and confined to the solid angle element $d\Omega$ along the unit vector $\mathbf{n} = \mathbf{k}/\omega$. Let ε be the radiation energy confined to volume L^3. Then, exploiting the definition of $I_{\mathbf{k}\lambda}$ and formula (28.13), we arrive at the following expression for the total radiation energy flux (up to the end of this section we will use ordinary units):

$$c\frac{\varepsilon}{L^3} = c\int_0^\infty \rho(\omega)\, d\omega = c\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\, \hbar\omega N_{\mathbf{k}\lambda} = \sum_\lambda \int I_{\mathbf{k}\lambda}\, d\omega\, d\Omega, \tag{28.33}$$

where we have introduced the spectral radiation-energy density $\rho(\omega)$. Allowing for the fact that $d^3k = c^{-3}\omega^2 d\omega d\Omega$, we arrive at a relationship that links $N_{\mathbf{k}\lambda}$ with $I_{\mathbf{k}\lambda}$:

$$N_{\mathbf{k}\lambda} = \left(\frac{2\pi c}{\omega}\right)^3 \frac{I_{\mathbf{k}\lambda}}{\hbar c}, \tag{28.34}$$

which is accompanied by a formula for the spectral density:

$$\rho(\omega) = \frac{\hbar\omega^3}{(2\pi c)^3} \sum_{\lambda} \int d\Omega N_{\mathbf{k}\lambda} = \frac{1}{c} \sum_{\lambda} d\Omega I_{\mathbf{k}\lambda}. \tag{28.35}$$

In the presence of external radiation (photons), along with photon emission in the $i \to f$ transition ($\varepsilon_i > \varepsilon_f$) there may be absorption of a photon from the external radiation in the inverse transition $f \to i$ with the amplitude (see Section 19)

$$\langle N_{\mathbf{k}\lambda} - 1;\ i|S|N_{\mathbf{k}\lambda};\ f\rangle = \sqrt{N_{\mathbf{k}\lambda}}\, S_{if}, \tag{28.36}$$

where $S_{if} = S_{fi}^*$ is the matrix element complex conjugate to the matrix element representing the spontaneous transition $i \to f$, (28.7). The probability of photon absorption is therefore proportional to the probability of spontaneous emission (in the inverse transition $i \to f$) and is equal to the probability of stimulated emission (see (28.32)):

$$w_{if}^{(\text{absorp})} = N_{\mathbf{k}\lambda} w_{fi} = w_{fi}^{(\text{stim})}. \tag{28.37}$$

From (28.31) and (28.37) follows a relation between the probabilities of emission and absorption of a photon of the ($\mathbf{k}$, λ) type:

$$\frac{w_{fi}^{(\text{emis})}(\mathbf{k}, \lambda)}{w_{if}^{(\text{absorp})}(\mathbf{k}, \lambda)} = \frac{N_{\mathbf{k}\lambda} + 1}{N_{\mathbf{k}\lambda}}. \tag{28.38}$$

This remarkable relationship was first discovered by A. Einstein in 1916 even before the advent of a consistent quantum theory of radiation. As is well known, it opened up the possibility for the creation of a new type of devices, known as lasers and masers, which are widely used in various fields of science and technology.

In conclusion we note that often other quantities, known as Einstein's probability coefficients, are introduced for $i \rightleftarrows f$ transitions instead of the transition rates, namely, A_{fi}, B_{if}, and B_{fi}, which are related thus:

$$w_{fi}^{(\text{spon})} \equiv w_{fi} = A_{fi}, \quad w_{fi}^{(\text{stim})} = \rho(\omega) B_{fi}, \quad w_{if}^{(\text{absorp})} = \rho(\omega) B_{if}, \tag{28.39}$$

where $\rho(\omega)$ is the spectral energy density in the external radiation, (28.35), at the frequency $\omega = \omega_{fi} = \varepsilon_i - \varepsilon_f$ ($\varepsilon_i > \varepsilon_f$).

The relationship between these coefficients follows from (28.37) and (28.35) and can easily be established if we assume that the incident radiation is isotropic and nonpolarized (or $N_{\mathbf{k}\lambda} = N(\omega)$, that is, does not depend on the directions specified by $\mathbf{k}$ and $\mathbf{e}_{(\lambda)}$):

$$B_{fi} = B_{if} = \frac{\pi^2 c^3}{\hbar\omega^3} A_{fi}. \tag{28.40}$$

29. Synchrotron Radiation (Quantum Theory)

The classical theory of synchrotron radiation (see Section 17) can be applied as long as the characteristic energy of the emitted photons, $\hbar\omega_c$ (where $\omega_c = (ceH/\varepsilon)\,(\varepsilon/mc^2)^3$ is the frequency (17.38) corresponding to the maximum in the synchrotron radiation spectrum), is low compared to the energy of a relativistic electron, $\varepsilon \gg mc^2$, that is, as long as the dimensionless parameter

$$\chi = \frac{H}{H_0}\,\frac{\varepsilon}{mc^2} = \frac{\hbar\omega_c}{\varepsilon} \tag{29.1}$$

is considerably less than unity. Here we have introduced the concept of the critical field strength

$$H_0 = m^2c^3/\hbar e, \tag{29.2}$$

which for the electron approximately amounts to 4.41×10^{13} G. Thus, the quantum effects manifest themselves in synchrotron radiation only for high electron energies, namely $\varepsilon \gtrsim mc^2\,(H_0/H)$.

The power of the radiated photons in the emission by electrons moving in a magnetic field is relativistically invariant (see Sections 16 and 28), and therefore can depend only on invariant parameters. In the general case of arbitrary values of the electron energy and the magnetic-field strength H, only two independent invariants can be constructed from the field-strength tensor $F^{\mu\nu}$ ($F^{21} = -F^{12} = H$) and the components of the 4-momentum p^μ of the electron:*

$$\chi = \frac{e}{m^3}\left[-(F_{\mu\nu}p^\nu)^2\right]^{1/2} = \frac{p_\perp}{m}\,\frac{H}{H_0}\,,$$
$$f = \frac{e}{m^2}\left[\frac{1}{2}\,|F_{\mu\nu}F^{\mu\nu}|\right]^{1/2} = \frac{H}{H_0}\,. \tag{29.3}$$

Here and in what follows we employ the system of units in which $\hbar = c = 1$. Note that parameter χ in the form (29.1) belongs entirely to the case of a purely transverse motion $p_z = 0$, when $\varepsilon = (p_\perp^2 + m^2)^{1/2} \simeq p_\perp$.

Thus, the radiated power must be a function of the two parameters in (29.3):

$$W = W(\chi, f). \tag{29.4}$$

At present the most intense pulsed magnetic fields obtained in laboratories are of about 10^7 G, which is considerably lower than the critical field $H_0 \sim 10^{13}$ G, so that under such conditions f is much

* In a magnetic field H parallel to the z axis, only three out of the four components of p^μ are conserved, namely, the energy $\varepsilon = p^0$, the transverse component $p_\perp = |\mathbf{p}_\perp|$, where $\mathbf{p}_\perp \perp \mathbf{H}$, and the longitudinal component $p_z = p^3$. Only these three components have direct physical meaning, are integrals of motion, and enter the invariant combinations (29.3).

less than unity. Let us assume that the transverse momentum of the electron is high ($p_\perp \gg m$). Then $f \ll \chi$ and we can put $f \simeq 0$ in (29.4):

$$W \simeq W(\chi, 0), \tag{29.5}$$

that is, under the specified conditions the radiated power is a function of a single parameter χ, which characterizes the magnitude of quantum effects in synchrotron radiation.

29.1 Solution of the Dirac Equation for an Electron in a Constant Uniform Magnetic Field

The system of coordinates and the gauge for the potential A^μ of the external magnetic field are selected in such a manner that

$$\mathbf{H} \parallel Oz, \quad A^\mu = (0, 0, xH, 0). \tag{29.6}$$

Then the Hamiltonian of the Dirac equation, (28.4), assumes the form*

$$\mathrm{H} = \alpha_1 \mathrm{p}_x + \alpha_2 (\mathrm{p}_y + eHx) + \alpha_3 \mathrm{p}_z + \rho_3 m, \tag{29.7}$$

which implies that the operators H, p_y and p_z are integrals of motion and that the time independent solutions to the Dirac equation must be sought in the form

$$\psi(t, \mathbf{r}) = N\tilde{\psi}(x) \exp(-iet + ip_y y + ip_z z), \tag{29.8}$$

where N is a normalization constant. Substituting (29.8) into Eq. (28.4) and allowing for (29.7) and the explicit form of the matrices α and $\rho_3 = \gamma^0$ (see (8.48) and (8.52)), we arrive at a system of linear equations for the components $\tilde{\psi}_i$ ($i = 1, 2, 3, 4$) of the bispinor $\tilde{\psi}$:

$$\begin{aligned}
(\varepsilon \mp m)\tilde{\psi}_{1,3} + i\sqrt{eH}\left(\frac{d}{d\eta} + \eta\right)\tilde{\psi}_{4,2} - p_z\tilde{\psi}_{3,1} = 0, \\
(\varepsilon \mp m)\tilde{\psi}_{2,4} + i\sqrt{eH}\left(\frac{d}{d\eta} - \eta\right)\tilde{\psi}_{3,1} + p_z\tilde{\psi}_{4,2} = 0,
\end{aligned} \tag{29.9}$$

where instead of x we have introduced the dimensionless variable

$$\eta = \sqrt{eH}\left(x + \frac{p_y}{eH}\right). \tag{29.10}$$

System (29.9) shows that

$$\tilde{\psi}(x) = \begin{pmatrix} C_1\varphi_1 \\ iC_2\varphi_2 \\ C_3\varphi_1 \\ iC_4\varphi_2 \end{pmatrix}, \tag{29.11}$$

* In this section e stands for the absolute value of the electron charge, or $-e < 0$.

where C_i are constant spin coefficients (see below). Substitution of (29.11) into (29.9) enables finding independent second-order equations for the functions φ_1 and φ_2:

$$\left[\frac{d^2}{d\eta^2} + (\lambda \mp 1 - \eta^2)\right] \varphi_{1,2}(\eta) = 0,$$
$$\lambda = (\varepsilon^2 - m^2 - p_z^2)/eH \equiv p_\perp^2/eH. \tag{29.12}$$

These equations coincide with the equation for the wave functions of a harmonic oscillator, whereby the eigenvalues of parameter λ that comply with the requirement that the squares of φ_1 and φ_2 be integrable are

$$\lambda = 2n, \quad n = 0, 1, 2, \ldots, \tag{29.13}$$

while the eigenfunctions are Hermite functions,

$$\varphi_1 = u_{n-1}(\eta),$$
$$\varphi_2 = u_n(\eta) = (2^n n! \pi^{1/2})^{-1/2} e^{-\eta^2/2} H_n(\eta), \tag{29.14}$$

where $H_n(\eta)$ are Hermite polynomials.

From (29.12) and (29.13) follows the formula for the energy spectrum of an electron moving in a constant uniform magnetic field:

$$\varepsilon_{n p_z} = [m^2 + 2eHn + p_z^2]^{1/2}. \tag{29.15}$$

We see that the energy levels are independent of p_y and the spin quantum number (see below), that is, are degenerate (infinitely degenerate, for that matter).

Let us now determine the spin coefficients C_i in (29.11). This can be done by requiring that ψ given by (29.8) be an eigenfunction of the spin polarization operator, which is an integral of motion. For such an operator we take the invariant spin operator*

$$\mathrm{M} = \frac{e}{2m^2} \tilde{F}_{\mu\nu} \mathrm{p}^\mu \mathrm{S}^\nu, \tag{29.16}$$

where, as usual, $\mathrm{p}^\mu = i\partial/\partial x_\mu$, $\tilde{F}_{\mu\nu} = (1/2)\, \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}$ is the dual field-strength tensor, and S^ν is the 4-vector spin operator defined thus:

$$\mathrm{S}^\mu = \gamma^5 \left(\gamma^\mu - \frac{1}{m} \mathrm{p}^\mu\right). \tag{29.17}$$

Here $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3 = -\rho_1$ and $\mathrm{P}^\mu = \mathrm{p}^\mu - eA^\mu$, where $\mathrm{P}^0 = \mathrm{H} - eA^0$, and H is the Hamiltonian (28.4). We wish to emphasize that the spin operators are defined on the class of functions obeying

* A detailed description of the various types of spin operators can be found in [2].

the Dirac equation (and therefore $p^0 = i\partial/\partial t = H$). The components of operator (29.17) may be specified as follows (see (21.11)):

$$S^{\nu} = \left(\frac{1}{m}\Sigma\cdot\mathbf{P},\ \rho_3\Sigma + \frac{1}{m}\rho_1\mathbf{P}\right), \tag{29.17a}$$

with the matrices Σ defined in (8.52).

We wish to find the explicit form of operator (29.16) in the case of a magnetic field with the nonzero components of $\widetilde{F}_{\mu\nu}$ being $\widetilde{F}_{03} = -\widetilde{F}_{30} = H$. In the reference frame in which the electron is at rest ($\mathbf{P} = 0$), the operator M describes the interaction between the spin magnetic moment of the electron with the magnetic field:

$$\mathrm{M}|_{\mathbf{P}=0} = -\mu_0\cdot\mathbf{H}, \tag{29.18}$$

with $\mu_0 = -(e/2m)\,\Sigma$. In the laboratory reference frame we have

$$\mathrm{M} = \frac{eH}{2m^2}\left(\mathrm{HS}_z - \mathrm{P}_z\frac{\Sigma\cdot\mathbf{P}}{m}\right). \tag{29.19}$$

It can easily be verified that $[H, S_z] = 0$ and $[H, \Sigma\cdot\mathbf{P}] = 0$, which when combined with (29.19), yields $[H, M] = 0$, that is, M is indeed an integral of motion.

For stationary states (29.8) we have

$$\mathrm{H} = \rho_1(\Sigma\cdot\mathbf{P}) + \rho_3 m = \varepsilon, \quad \mathrm{P}_z = p_z, \quad \Sigma\cdot\mathbf{P} = \rho_1\varepsilon + i\rho_2 m,$$

where we have allowed for the fact that $\alpha = \rho_1\Sigma$ and $\rho_1\rho_3 = -i\rho_2$. Substituting these expressions and S_z (from (29.17a)) into (29.19) yields

$$\mathrm{M} = \frac{eH}{2m^2}(\varepsilon\rho_3\Sigma_3 - ip_z\rho_2). \tag{29.20}$$

This together with $\rho_3^2 = \rho_2^2 = 1$ and $\rho_2\rho_3 + \rho_3\rho_2 = 0$ yields

$$\mathrm{M}^2 = \left(\frac{eH}{2m^2}\right)^2\varepsilon_\perp^2,$$

where $\varepsilon_\perp = (\varepsilon^2 - p_z^2)^{1/2} = (m^2 + 2eHn)^{1/2}$ Thus, the eigenvalues M of operator M are

$$M = \frac{eH}{2m^2}\varepsilon_\perp\zeta, \tag{29.21}$$

where $\zeta = \pm 1$ is the spin quantum number.

Let us require that the wave function (29.8) be an eigenfunction of M, that is,

$$\mathrm{M}\psi = \zeta\frac{eH}{2m^2}\varepsilon_\perp\psi. \tag{29.22}$$

Substituting operator M in the form (29.20) and allowing for (29.11), we arrive at a system of equations for the spin coefficients C_i:

$$(\varepsilon \mp \zeta\varepsilon_\perp) C_{1,3} = p_z C_{3,1}, \quad (\varepsilon \pm \zeta\varepsilon_\perp) C_{2,4} = -p_z C_{4,2}. \quad (29.23)$$

This must be solved together with the system obtained through substituting (29.11) into (29.9) via the recurrence formula for the Hermite functions (29.14), namely,

$$\left(\frac{d}{d\eta} + \eta\right) u_n = \sqrt{2n}\, u_{n-1}, \quad \left(\frac{d}{d\eta} - \eta\right) u_{n-1} = -\sqrt{2n} u_n. \quad (29.24)$$

This other system has the form

$$\begin{aligned} (\varepsilon \mp m)\, C_{1,3} - p_\perp C_{4,2} - p_z C_{3,1} = 0, \\ (\varepsilon \mp m)\, C_{2,4} - p_\perp C_{3,1} + p_z C_{4,2} = 0, \end{aligned} \quad (29.25)$$

where

$$p_\perp = \sqrt{2eHn} = \sqrt{\varepsilon^2 - p_z^2 - m^2}.$$

Combining (29.23) with (29.25) yields the following formulas for the spin coefficients:

$$C_{1,3} = -\frac{1}{2\sqrt{2}} \left(1 + \zeta \frac{m}{\varepsilon_\perp}\right)^{1/2} \left[\left(1 + \frac{p_z}{\varepsilon}\right)^{1/2} \pm \zeta \left(1 - \frac{p_z}{\varepsilon}\right)^{1/2}\right], \quad (29.26)$$

$$C_{2,4} = \frac{1}{2\sqrt{2}} \zeta \left(1 - \zeta \frac{m}{\varepsilon_\perp}\right)^{1/2} \left[\zeta \left(1 - \frac{p_z}{\varepsilon}\right)^{1/2} \mp \left(1 + \frac{p_z}{\varepsilon}\right)^{1/2}\right],$$

which are normalized by the condition

$$\sum_{i=1}^{4} |C_i|^2 = 1. \quad (29.26a)$$

The value of the general normalization constant N in (29.8) can be found from the normalization condition

$$\int_{-\infty}^{\infty} dx \int_{-L/2}^{L/2} dy \int_{-L/2}^{L/2} dz \psi^+ \psi = 1 \quad (29.27)$$

and proves to be

$$N = (eH)^{1/4}/L, \quad (29.28)$$

where we have introduced the normalization length L along the y and z axes and have allowed for the fact that the Hermite functions (29.14) are normalized in variable η (see (29.10)) to unity.

Combining (29.8), (29.11), (29.14), and (29.18), we arrive at the final form of the wave function of an electron moving in a uniform magnetic field:

$$\psi_{n p_y p_z \zeta}(t, \mathbf{r}) = \frac{(eH)^{1/4}}{L} \exp(-i\varepsilon_{n p_z} t + i p_y y + i p_z z) \begin{pmatrix} C_1 u_{n-1}(\eta) \\ iC_2 u_n(\eta) \\ C_3 u_{n-1}(\eta) \\ iC_4 u_n(\eta) \end{pmatrix}, \tag{29.29}$$

which function is specified by four quantum numbers, n, p_y, p_z, and ζ. The four quantum numbers have the following meaning: $n = 0, 1, 2, \ldots$ is the principal quantum number, which determines the magnitude of the transverse momentum $p_\perp$ and, hence, the radius R of the quasi-classical orbit (for $n \gg 1$),

$$p_\perp = \sqrt{2eHn}, \quad R = \frac{p_\perp}{eH} = \left(\frac{2n}{eH}\right)^{1/2}, \tag{29.30}$$

p_z is the projection of the momentum vector on the direction of H; p_y specifies the coordinate of the orbit's center (see (29.10)),

$$x_0 = -p_y/eH; \tag{29.31}$$

and the spin number ζ specifies the state with the spin oriented along ($\zeta = +1$) or in opposition to ($\zeta = -1$) H, that is, a state with transverse polarization (with respect to the momentum vector; this term has its origin in the fact that at $p_z = 0$ the spin polarization operator M given by (29.20) is proportional to $\Sigma_3 = (\Sigma \cdot \mathbf{H})/H$).

29.2 The Quasi-Classical Solution to the Dirac Equation

Using the exact wave functions (29.29), we can find the expression for the radiated power in synchrotron radiation valid for all values of electron energy and magnetic-field strength, that is, for arbitrary values of χ and f (see [2]). However, an analysis of this expression for the general case presents unsurmountable difficulties. We will restrict our discussion to the important case that is realized in laboratory conditions (in accelerators and storage rings), namely, the electron energies are ultrarelativistic ($p_\perp \gg m$) and the magnetic field is weak, $H \ll H_0$, that is $f \ll 1$ and $f \ll \chi$ (see (29.3) and (29.5)). Under such conditions both the initial state and the final state (after emission of electron) of the electron are characterized by large quantum numbers $n \gg 1$, that is, are quasi-classical. Therefore, even when χ is of the order of unity (see (29.1) and (29.3)), that is, when quantum effects become apparent in the radiation, it has proved possible to introduce the notion of the quasi-classical electron trajectory prior

to and after the emission act. The reasoning of Section 17 that concerns the character of radiation formation on an extremely small section of the electron path (see (17.20)), on which section the electron momentum rotates through an angle of the order of $m/\varepsilon \ll 1$, is still valid under these conditions. Both the quasi-classicality condition $n \gg 1$, or in equivalent form (see (29.30))

$$p_\perp \gg (eH)^{1/2}, \tag{29.32}$$

and the small size of the region where radiation formation takes place make it possible, in calculating the probability and radiated power of synchrotron radiation, to employ the quasi-classical approximation directly in the electronic wave functions. This method has proved to be simpler than the common method of finding the quasi-classical asymptotic behavior of the exact matrix elements [2].

Let us find the quasi-classical asymptotic behavior of the exact wave function (29.29). We start with the differential equation (29.12), which is more conveniently written in the form

$$\left(\frac{d^2}{d\xi^2} + 2\frac{p_\perp}{\sqrt{eH}}\xi \mp 1 - \xi^2\right)\varphi_{1,2} = 0, \tag{29.33}$$

where instead of η given in (29.10) we have introduced a new dimensionless variable

$$\xi = (eH)^{1/2}\bar{x}, \quad \bar{x} = x - x_0 + R, \tag{29.34}$$

and x_0 and R are defined as in (29.30) and (29.31). By virtue of the quasi-classicality condition (29.32) parameter $p_\perp/(eH)^{-1/2}$ in Eq. (29.33) is large, and we can ignore the last term, $-\xi^2$. As a result, Eq. (29.33) is reduced to the equation for the Airy function Φ (see (17.34) and (17.34a)), so that we obtain the following asymptotic form of the solution to Eq. (29.33):

$$\varphi_k = \Phi(-x_k),$$

$$\Phi(-x_k) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} dt' \exp\left[-ix_k t' + i\frac{t'^3}{3}\right], \tag{29.35}$$

$$x_k = (2eHp_\perp)^{1/3}\left[\bar{x} + \frac{(-1)^k}{2p_\perp}\right], \quad k = 1, 2.$$

The physical meaning of the solution obtained in this manner becomes clear if in Eq. (29.33) we go over to variable $\bar{x}$. The function φ_k in (29.35) satisfies the equation without the term $\bar{x}^2$ and therefore serves as an approximation to the solution in the region where $\bar{x}^2 \ll R^2$ and, hence, describes the motion of the electron on a small section of the quasi-classical trajectory. It is this solution that is

especially convenient for studying problems involving radiation, since, as noted above, the radiation emitted by a relativistic quasi-classical electron is formed within a small section of the electron trajectory proportional to $(m/\varepsilon)\, R \ll R$.

The functions (29.35) are not quadratically integrable (in contrast to the exact functions (29.14)):

$$J_k = \int_{-\infty}^{\infty} \varphi_k^2\, dx$$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \exp\left[\frac{i}{3}(t'^3 - t''^3)\right] \int_{-\infty}^{\infty} dx \exp\left[-ix_k(t' - t'')\right]$$

$$= \frac{1}{2}(2eHp_\perp)^{-1/3} \lim_{L_0 \to \infty} \int_{-L_0/2}^{L_0/2} dt', \tag{29.36}$$

where we allowed for the fact that the inner integral with respect to x (see the relationship between x_k and x in (29.35)) is equal to $(2eHp_\perp)^{-1/3} 2\pi\delta(t' - t'')$. The fact that the integral in (29.36) is divergent is inessential, since the emission probability (emission rate) is expressed in terms of the matrix elements calculated, in the accepted approximation, using the φ_k functions; these matrix elements prove to be finite (see Eq. (29.44) below) by virtue of the small size of the region where the radiation is formed (it is inside this region that the asymptotic solution (29.35) holds true).

Just as in the case of normalization of plane waves along the y and z axes (see (29.8)), we can assume that the auxiliary quantity L_0 is finite (i.e. we introduce a finite "volume" L_0 for variable t' in the integral representation of φ_k in (29.35)), and this auxiliary quantity proves to be proportional to the normalization length L along the y axis. Indeed, let us take the integral

$$\tilde{J}_k = \int_{-\infty}^{\infty} \Phi(-x_k)\, \Phi(-x_k')\, dx, \tag{29.37}$$

where x_k' is obtained from x_k of (29.35) by replacing p_y with $p_y' \neq p_y$. Performing calculations similar to (29.36), we find that

$$\tilde{J}_k = \frac{1}{2b} \int_{-\infty}^{\infty} dt' \exp\left[i\,(b/eH)\,(p_y - p_y')\,t'\right] = \frac{\pi eH}{b^2}\, \delta(p_y - p_y'), \tag{29.37a}$$

where we have introduced the notation

$$b = (2eHp_\perp)^{1/3}. \tag{29.37b}$$

Comparison of (29.36) with (29.37a) shows that at $p_y' = p_y$ we must have $\widetilde{J}_k = J_k$, which immediately yields

$$J_k = \frac{1}{2b} L_0 = \frac{\pi eH}{b^2} \frac{L}{2\pi}, \quad L_0 = \frac{eH}{b} L, \tag{29.38}$$

since we must assume that $\delta(0) = \delta(p_y - p_y')|_{p_y'=p_y}$ is equal, as usual, to $L/2\pi$ ($L_y \equiv L$ is the normalization length along the y axis).

Subjecting now the total wave function to normalization condition (29.27), we arrive, after we allow for (29.36) and (29.38), at the normalized quasi-classical wave function in the form

$$\psi_{p_\perp p_y p_z \zeta}(t, \mathbf{r}) = L^{-3/2} \left(\frac{2b^2}{eH}\right)^{1/2} \exp(-i\varepsilon t + ip_y y + ip_z z) \begin{pmatrix} C_1 \Phi(-x_1) \\ iC_2 \Phi(-x_2) \\ C_3 \Phi(-x_1) \\ iC_4 \Phi(-x_2) \end{pmatrix}, \tag{29.39}$$

where the energy $\varepsilon = (m^2 + p_\perp^2 + p_z^2)^{1/2}$, $b = (2eHp_\perp)^{1/2}$, and $\Phi(-x_k)$ ($k = 1, 2$) and C_i ($i = 1, 2, 3, 4$) are defined in (29.35) and (29.26).

29.3 Emission Probability

The probability of an electron transferring from state $|i\rangle = |p_\perp, p_y, p_z, \zeta\rangle$ to state $|f\rangle = |p_\perp', p_y', p_z', \zeta'\rangle$ and simultaneously emitting a photon with 4-momentum $k = (\omega, \mathbf{k})$ and polarization $e_{(\lambda)}$ is determined through the general formula (28.14). The two independent linear polarization 4-vectors of the photon can be specified in a unified manner in covariant form for an arbitrary constant field $F^{\mu\nu}$ thus:

$$e_{(1)}^\mu = \frac{F^{\mu\nu} k_\nu}{[-(F^{\alpha\beta} k_\beta)^2]^{1/2}}, \quad e_{(2)}^\mu = \frac{\widetilde{F}^{\mu\nu} k_\nu}{[-(F^{\alpha\beta} k_\beta)^2]^{1/2}}, \tag{29.40}$$

where $\widetilde{F}^{\mu\nu}$ is the dual field-strength tensor. These vectors (irrespective of the concrete form of $F^{\mu\nu}$) prove to be unit vectors that are four-dimensional transverse and orthogonal to each other:

$$e_{(1)}^2 = e_{(2)}^2 = -1, \quad e_{(1)} e_{(2)} = e_{(1)} k = e_{(2)} k = 0. \tag{29.41}$$

Under a gauge transformation of the (16.57) type these vectors are transformed into the common three-dimensional transverse vectors of the σ- and the π-component of polarization (see (17.4)):

$$\begin{aligned} e_{(1)}^\mu &= (0, \mathbf{e}_\sigma), \quad \mathbf{e}_\sigma = (-\sin\varphi, \cos\varphi, 0), \\ e_{(2)}^\mu &= (0, \mathbf{e}_\pi), \quad \mathbf{e}_\pi = (-\cos\theta\cos\varphi, -\cos\theta\sin\varphi, \sin\theta), \end{aligned} \tag{29.42}$$

where θ and φ are the spherical angles of the photon momentum vector

$$\mathbf{k} = \omega (\sin\theta\cos\varphi, \quad \sin\theta\sin\varphi, \quad \cos\theta). \tag{29.43}$$

In the gauge (29.42), the problem is reduced to calculating the matrix elements $\mathbf{j}_{fi}$ (28.14a) with the wave functions (29.39). Integration with respect to y and z yields delta functions, which together with $\delta(\omega - \omega_{fi})$ represent the conservation laws

$$\varepsilon = \varepsilon' + \omega, \quad p_y = p_y' + k_y, \quad p_z = p_z' + k_z. \tag{29.44}$$

This together with (29.39) yields

$$\begin{gathered}\mathbf{j}_{fi} = \frac{1}{L^3}\frac{2bb'}{eH}(2\pi)^2\delta(p_y' + k_y - p_y)\,\delta(p_z' + k_z - p_z)\,\bar{\alpha}, \\ \bar{\alpha} = \int\limits_{-\infty}^{\infty} dx e^{-ik_x x}\Phi_f^{\dagger}(x)\,\alpha\Phi_i'(x),\end{gathered} \tag{29.45}$$

where $\Phi_i(x)$ stands for the column consisting of Airy functions in (29.39), while $\Phi_f^{\dagger}(x)$ is a similar row for the electron in the final state, and $b' = (2eHp_{\perp}')^{1/3}$.

Calculation of the matrix elements of $\bar{\alpha}$ is reduced to evaluating the integrals ($i, k = 1, 2$)

$$\begin{aligned}J_{ik} &= \int\limits_{-\infty}^{\infty} dx e^{-ik_x x}\Phi[-b'(x + a_i')]\,\Phi[-b(x + a_k)] \\ &= \frac{\sqrt{\pi}\,e^{i\Omega_{ik}}}{(b^3 - b'^3)^{1/3}}\Phi(z_{ik}),\end{aligned} \tag{29.46}$$

$$z_{ik} = \frac{bb'}{(b^3 - b'^3)^{1/3}}\left(a_k - a_i' - \frac{k_x^2}{b^3 - b'^3}\right),$$

$$\Omega_{ik} = -\frac{k_x^2}{3}\frac{b^3 + b'^3}{(b^3 - b'^3)^2} + k_x\frac{b^3 a_k - b'^3 a_i'}{b^3 - b'^3},$$

$$a_k = \frac{p_y + p_{\perp}}{eH} + \frac{(-1)^k}{2p_{\perp}},$$

with the quantities with primes referring to the final state of the electron. Formula (29.46) can easily be obtained by employing the integral representation (29.35).

Taking into account the explicit form of α (see (8.52)) and the wave functions (29.39), we can express $\bar{\alpha}$ in (29.45) in terms of the inte-

grals J_{ik} and the spin coefficients C_n and C'_n ($n = 1, 2, 3, 4$) of the initial and final states:

$$\begin{pmatrix} -i\bar{\alpha}_1 \\ \bar{\alpha}_2 \end{pmatrix} = (C'_1C_4 + C'_3C_2)\,J_{12} \mp (C'_2C_3 + C'_4C_1)\,J_{21},$$
$$\bar{\alpha}_3 = (C'_1C_3 + C'_3C_1)\,J_{11} - (C'_2C_4 + C'_4C_2)\,J_{22}. \tag{29.47}$$

Substituting (29.45) into the general formula (28.14a) and summing over the final states of the electron in accord with

$$\sum_{p'_\perp,\, p'_y,\, p'_z} = \left(\frac{L}{2\pi}\right)^3 \int_0^\infty dp'_\perp \int_{-\infty}^{\infty} dp'_y \int_{-\infty}^{\infty} dp'_z, \tag{29.48}$$

we obtain the differential probability of emission (differential emission rate) of a photon with 4-momentum $k = (\omega, \mathbf{k})$ and polarization $\mathbf{e}_{(\lambda)}$ in the following form:

$$dw_\lambda = \frac{e^2}{(2\pi)^2 L}\,\frac{d^3k}{\omega} \int_0^\infty dp'_\perp \delta(\omega + \varepsilon' - \varepsilon) \left(\frac{2bb'}{eH}\right)^2 |\mathbf{e}^*_{(\lambda)} \cdot \bar{\boldsymbol{\alpha}}|^2, \tag{29.49}$$

where $\varepsilon' = (m^2 + p'^2_\perp + p'^2_z)^{1/2}$, $p'_z = p_z - k_z$, and $p'_y = p_y - k_y$ (see (29.44)). In deriving (29.49) we have allowed for the fact that $[\delta(p'_y + k_y - p_y)\,\delta(p'_z + k_z - p_z)]^2 = (L/2\pi)^2\delta(p'_y + k_y - p_y)\,\delta(p'_z + k_z - p_z)$. Integration with respect to $p'_\perp$ is carried out using the delta function, formula (12.11), and the relationship $\partial\varepsilon'/\partial p'_\perp = p'_\perp/\varepsilon'$:

$$\int dp'_\perp \delta(\omega + \varepsilon' - \varepsilon) = \frac{\varepsilon - \omega}{p'_\perp}, \tag{29.50}$$

where $p'_\perp$ in the denominator is the (positive) root of the equation $\omega + \varepsilon' - \varepsilon = 0$.

Let us transform the phase volume element $d^3k = \omega^2 d\omega \sin\theta\, d\theta\, d\varphi$ (see (29.43)) with due regard for the fact that in the given ultrarelativistic case ($p'_\perp \gg m$) and that at $p_z = 0$ the main contribution to the integrals with respect to the angles

$$\theta = \frac{\pi}{2} - \psi, \quad \varphi = \delta - \frac{\pi}{2} \tag{29.51}$$

in the total emission rate is provided by the region where $|\psi| \ll 1$ and $|\delta| \ll 1$, just as in the classical case (see Section 17). Instead of ω and ψ we introduce new relativistically invariant variables

$$u = \frac{p_\perp - p'_\perp}{p'_\perp} = \frac{\chi - \chi'}{\chi'} \simeq -\frac{\omega}{p_\perp - \omega},$$
$$\tau = \frac{p_\perp}{m}\psi = -e\frac{\tilde{F}_{\mu\nu}p'^\mu p^\nu}{m^4(\chi - \chi')}, \tag{29.52}$$

where $\chi' = Hp'_\perp/H_0 m = (e/m^3)\,[-(F_{\mu\nu}p'^{\nu})^2]^{1/2}$ (see (29.3)). Then the formula for the volume element d^3k assumes the following form (see (29.51)):

$$d^3k = \omega^2\, d\omega\, d\psi\, d\delta = mp_\perp^2\, \frac{u^2\, du}{(1+u)^4}\, d\tau\, d\delta. \tag{29.53}$$

In the accepted approximation, the arguments z_{ik} and the phases Ω_{ik} in (29.46) (where we allow only for the leading terms in the expansion in powers of $m/p_\perp \ll 1$) are

$$z_{ik} = \left(\frac{u}{2\chi}\right)^{2/3}\left[1+\tau^2-(-1)^i f\left(\frac{2+u}{u}-\frac{2}{u}\,\delta_{ik}\right)\right],$$
$$\Omega_{ik} = \Omega_0 + (-1)^i\,(\delta_{ik}-1)\,\frac{u}{1+u}\,\delta, \tag{29.54}$$

where Ω_0 is an inessential term that depends neither on i nor on k (the general phase factor $\exp(i\Omega_0)$ in the matrix elements (29.47) contributes nothing to the emission rate (29.49)).

Now let us consider in greater detail the region of variable t' that contributes the most to the formation of the Airy functions $\Phi(z_{ik})$ or the integrals J_{ik} (29.46), which enter into the matrix elements (29.47). Using (29.35), we find that

$$J_{ik} = \frac{1}{2}\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\,\delta(b't''-bt'-k_x)$$
$$\times \exp\left\{ib'a_i't''-\frac{i}{3}\,t''^3-iba_kt'+\frac{i}{3}\,t'^3\right\}. \tag{29.55}$$

The integral with respect to t'' is evaluated via the delta function, so that we are left with the integral with respect to t', with the region near points $t'_{1,2}$ of stationary phase (the argument of the exponential in (29.55), where $t'' = (bt' + k_x)/b'$) contributes the most to the value of this integral. The points $t'_{1,2}$ are the roots of the equation obtained by nullifying the derivative with respect to t' of the argument of the exponential:

$$(b^3 - b'^3)\,t'^2 + 2b^2k_xt' + b\,[k_x^2 + b'^3\,(a_k - a_i')] = 0, \tag{29.56}$$

which yields

$$t'_{1,2} = -\frac{b^2}{b^3-b'^3}\,k_x \pm \frac{b'}{(b^3-b'^3)^{1/2}}\left[bb'\left(a_k-a_i'-\frac{k_x^2}{b^3-b'^3}\right)\right]^{1/2},$$

or, after allowing for (29.46), (29.37b), and (29.52) and the fact that $k_x \simeq \omega\delta$,

$$t'_{1,2} = -\left(\frac{p_\perp^2}{2eH}\right)^{1/3}\delta \pm i\,\frac{(z_{ik})^{1/2}}{u^{1/3}}. \tag{29.57}$$

Thus, because z_{ik} is independent of δ (see (29.54)), we arrive at the relation

$$d\delta = \left(\frac{2eH}{p_\perp^2}\right)^{1/3} dt', \tag{29.58}$$

or (see (29.38))

$$\int d\delta = \left(\frac{2eH}{p_\perp^2}\right)^{1/3} L_0 = \frac{eH}{p_\perp} L. \tag{29.58a}$$

Note that (29.57) expresses the fact that the radiation in a given direction is gathered from a small section of the quasi-classical electron trajectory (see also Section 17).

Next we allow for the fact that the magnetic field is weak, $H \ll H_0$, that is, $f = H/H_0 \ll 1$. We can therefore expand $\Phi(z_{ik})$ in powers of parameter f (see (29.54)):

$$\Phi(z_{ik}) \simeq \Phi(z) - (-1)^i f \left(\frac{2+u}{u} - \frac{2}{u}\delta_{ik}\right)\left(\frac{u}{2\chi}\right)^{2/3} \Phi'(z), \tag{29.59}$$
$$z = \left(\frac{u}{2\chi}\right)^{2/3} (1+\tau^2).$$

In calculating the matrix elements (29.47) we expand the spin coefficients C_i and C_i' (see (29.26)) in powers of the small quantities $m/\varepsilon_\perp$, $m/\varepsilon_\perp'$, and p_z'/ε' and leave only the leading terms. Substituting the found expressions for $\overline{\alpha}_i$ into (29.49) and allowing for (29.53) and (29.59), we arrive, after integrating with respect to angle δ via (29.58a) (the quantity $|\, e_{(\lambda)}^* \cdot \overline{\alpha} \,|^2$ in (29.49) proves to be independent of δ in view of the axial symmetry of the problem), at the following expression for the angular spectral distribution of the emission rate written in terms of the invariant variables u and τ defined in (29.52):

$$\begin{pmatrix} dw_\sigma \\ dw_\pi \end{pmatrix} = \frac{e^2}{2\pi} \frac{m^2}{p_\perp} \left(\frac{u}{2\chi}\right)^{1/3} \frac{du}{(1+u)^3}\, d\tau$$
$$\times \left\{ \frac{1+\zeta\zeta'}{2} \left[\left((2+u)\left(\frac{2\chi}{u}\right)^{1/3} \Phi' + \zeta u \Phi\right)^2 \right] \right.$$
$$\left. + \frac{1-\zeta\zeta'}{2} \left[\begin{matrix} u^2\tau^2\Phi^2 \\ u^2\left(\left(\frac{2\chi}{u}\right)^{1/3} \Phi' - \zeta\Phi\right)^2 \end{matrix} \right] \right\}, \tag{29.60}$$
$$\Phi = \Phi(z), \quad \Phi' = d\Phi(z)\, dz, \quad z = \left(\frac{u}{2\chi}\right)^{2/3} (1+\tau^2).$$

Here the labels σ and π stand for the σ- and the π-component of the radiation's polarization (see (29.42)). In addition, the emission rate is split into two terms, one describing emission without spin-flip of the electron ($\zeta = \zeta'$) and the other with spin-flip ($\zeta' = -\zeta$).

29.4 Radiated Power

The radiated power of synchrotron radiation is obtained by multiplying the emission rate (29.60) into the photon energy $\omega = up_\perp/(1+u)$ (see (29.52)):

$$dW = p_\perp \frac{u}{1+u}\, dw. \tag{29.61}$$

We will restrict our discussion to the question of radiated power (in synchrotron radiation) summed over the photon polarizations and the final spin states of the electron and averaged over the initial states. Applying all this to (29.60) and substituting the resulting expression into (29.61), we arrive at the following formula for the angular spectral distribution of the radiated power in synchrotron radiation:

$$\begin{aligned}\frac{dW}{d\tau\, du} &= \frac{2}{\pi} e^2 m^2 \left(\frac{u}{2\chi}\right)^{1/3} \frac{u}{(1+u)^3} \\ &\times \left\{\left[\tau^2 + \frac{u^2}{2(1+u)}(1+\tau^2)\right]\Phi^2(z)\right. \\ &\left. + \left(\frac{2\chi}{u}\right)^{2/3}\left[1 + \frac{u^2}{2(1+u)}\right]\Phi'^2(z)\right\}, \\ z &= \left(\frac{u}{2\chi}\right)^{2/3}(1+\tau^2).\end{aligned} \tag{29.62}$$

Let us compare (29.62) with a similar classical expression (see (17.36)). Note that in common units (at $p_z = 0$) we have (see (29.52) and (29.1))

$$u = \frac{\hbar\omega}{\varepsilon - \hbar\omega}, \quad \chi = \frac{\hbar\omega_c}{\varepsilon}, \tag{29.63}$$

where $\varepsilon \simeq p_\perp$ is the electron energy, and ω_c is the critical frequency (in the classical meaning of the term; see (17.38)).

In the classical limit ($\hbar\omega \ll \varepsilon$, that is, electron recoil during photon emission is small) we have

$$u \simeq \hbar\omega/\varepsilon, \; u/\chi \simeq \omega/\omega_c, \tag{29.63a}$$

and formula (29.62) is transformed into the classical formula (see (17.36)).

The power spectrum of synchrotron radiation can be found by integrating (29.62) with respect to the angular variable τ via the formulas

$$\int_{-\infty}^{\infty} d\tau \Phi^2(z) = \sqrt{\pi}\, 2^{-2/3} x^{-1/2} \Phi_1(x),$$

$$\int_{-\infty}^{\infty} d\tau \tau^2 \Phi^2(z) = -\frac{\sqrt{\pi}}{4} 2^{1/3} x^{-3/2} [\Phi'(x) + x\Phi_1(x)], \quad (29.64)$$

$$\int_{-\infty}^{\infty} d\tau \Phi'^2(z) = -\frac{\sqrt{\pi}}{8} 2^{2/3} x^{-1/2} [3\Phi'(x) + x\Phi_1(x)],$$

where

$$x = \left(\frac{u}{\chi}\right)^{2/3}, \quad \Phi_1(x) = \int_x^{\infty} \Phi(y)\, dy. \quad (29.64a)$$

Using (29.64) and (29.62), we can obtain the synchrotron-radiation spectrum in the form

$$\frac{dW}{du} = -\frac{e^2 m}{\sqrt{\pi}} \frac{u}{(1+u)^3} \left\{\Phi_1(x) + \frac{2}{x}\left[1 + \frac{u^2}{2(1+u)}\right]\Phi'(x)\right\}, \quad (29.65)$$

$$x = (u/\chi)^{2/3}.$$

In the classical limit (see (29.63a)) this formula coincides with the classical expression for the synchrotron-radiation spectrum, (17.45). The asymptotic behavior of the spectrum can be found by exactly the same means as in classical theory (see Section 17). Allowing for (17.46), we find that for $u \ll \chi$

$$\frac{dW}{du} \simeq \frac{3^{1/6}}{\pi} \Gamma(2/3)\, e^2 m \frac{u}{(1+u)^3} \left(\frac{\chi}{u}\right)^{2/3} \left[1 + \frac{u^2}{2(1+u)}\right], \quad (29.66)$$

while for $u \ll 1$ ($\hbar\omega \ll \varepsilon$) this formula yields $dW/du \propto u^{1/3}$ and transforms into the classical formula (17.47). For $u \gg \chi$ ($x \gg 1$), we combine (17.48) with (29.65) and find that

$$\frac{dW}{du} \simeq \frac{e^2 m}{2\sqrt{\pi}} \frac{u}{(1+u)^3} \left(\frac{\chi}{u}\right)^{1/2} \left[1 + \frac{u^2}{1+u}\right] \exp\left(-\frac{2}{3}\frac{u}{\chi}\right), \quad (29.67)$$

which for $u \ll 1$ leads to the classical formula (17.49) and for $u \gg 1$ ($\hbar\omega \simeq \varepsilon$, that is, the photon carries away almost the entire energy of the electron), to

$$\frac{dW}{du} \simeq \frac{e^2 m}{2\sqrt{\pi}} \frac{1}{u} \left(\frac{\chi}{u}\right)^{1/2} \exp\left(-\frac{2}{3}\frac{u}{\chi}\right). \quad (29.67a)$$

The maximum in the distribution in u lies at $u \sim \chi$ (this follows directly from a comparison of (29.66) with (29.67) or from the exact formula (29.65) in which the argument of the Airy function, x, must be of the order of unity in the vicinity of the maximum). Allowing for the fact that $u = \hbar\omega/(\varepsilon - \hbar\omega)$, we arrive at the following estimate for the position of the maximum, ω_{max}, in the synchrotron-radiation spectrum:

$$\hbar\omega_{max} \sim \frac{\varepsilon\chi}{1+\chi} \simeq \begin{cases} \varepsilon\chi = \hbar\omega_c \text{ if } \chi \ll 1, \\ \varepsilon \text{ if } \chi \gg 1. \end{cases} \tag{29.68}$$

The energy of the electron in the final state, $\varepsilon' = \varepsilon/(1+u)$, in the region of maximum radiated power of synchrotron radiation can be estimated at

$$\varepsilon' \sim \begin{cases} \varepsilon \text{ if } \chi \ll 1, \\ \varepsilon/\chi \text{ if } \chi \gg 1. \end{cases} \tag{29.68a}$$

The reader can see that the electron in the final state remains ultrarelativistic (only in this case can we apply the method of synchrotron-radiation calculations based on the use of quasi-classical wave functions (29.39)), provided that $\varepsilon/\chi \gg m$, or (see (29.1))

$$f = H/H_0 \ll 1, \tag{29.69}$$

which is exactly what was assumed above (see (29.5) and (29.59)). Thus, the method is self-consistent, as it should be.

Let us express the synchrotron-radiation spectrum (29.65) in terms of the MacDonald functions via (17.50) and (17.51) by introducing a new spectral variable y and a new parameter ξ thus:

$$y = \frac{2}{3}\frac{u}{\chi}, \quad \xi = \frac{3}{2}\chi. \tag{29.70}$$

As a result we find that

$$\frac{dW}{dy} = W_{cl}F(y),$$
$$F(y) = \frac{9\sqrt{3}}{8\pi}\frac{y}{(1+\xi y)^3}\left[\int_y^\infty K_{5/3}(x)\,dx + \frac{\xi^2 y^2}{1+\xi y}K_{2/3}(y)\right], \tag{29.71}$$

with

$$W_{cl} = \frac{2}{3}\frac{ce^2}{R^2}\left(\frac{\varepsilon}{mc^2}\right)^4 = \frac{2}{3}cr_0^2H^2\left(\frac{\varepsilon}{mc^2}\right)^2 \tag{29.72}$$

the classical radiated power in synchrotron radiation (see (17.43)), and $r_0 = e^2/mc^2$ the classical electron radius (here we use common units). As $\xi \to 0$, the function $F(y)$ transforms into the classical function $f(y)$ (see (17.52)).

Let us take up the case $\xi \ll 1$, that is, low energies $\varepsilon \ll m\,(H_0/H)$. Then in (29.71) we can carry out an expansion in powers of ξ, which corresponds to an expansion in powers of Planck's constant $\hbar$. To within terms of the order of $\xi^2 \sim \hbar^2$, the following formula for the synchrotron-radiation spectrum is valid:

$$F(y) = \frac{9\sqrt{3}}{8\pi}\, y\left[(1 - 3\xi y + 6\xi^2 y^2)\int\limits_y^\infty K_{5/3}(x)\,dx + \xi^2 y^2 K_{2/3}(y)\right]. \tag{29.73}$$

The total radiated power is obtained by integrating over the spectrum (29.71):

$$W = W_{\mathrm{cl}} \int\limits_0^\infty F(y)\,dy. \tag{29.74}$$

For $\xi \ll 1$, using expansion (29.73), the formula

$$\int\limits_0^\infty y^{q-1} K_p(y)\,dy = 2^{q-2}\Gamma\left(\frac{q-p}{2}\right)\Gamma\left(\frac{q+p}{2}\right), \tag{29.75}$$

and (29.74), we can find the quantum correction terms to the classical expression for the radiated power in synchrotron radiation:

$$\begin{aligned} W &= W_{\mathrm{cl}}\left(1 - \frac{55\sqrt{3}}{24}\xi + \frac{64}{3}\xi^2 + \ldots\right) \\ &= W_{\mathrm{cl}}\left(1 - \frac{55}{16}\sqrt{3}\chi + 48\chi^2 + \ldots\right). \end{aligned} \tag{29.76}$$

Thus, for $\chi \ll 1$ the quantum effects lead only to small corrections to the classical spectrum and radiated power of synchrotron radiation (see (29.73) and (29.76)). In modern electron synchrotrons, the magnetic field H is of the order of 10^4 G and the quantum effects may become noticeable when $\varepsilon \gtrsim mc^2\,(H_0/H) \sim 10^3$ GeV. This condition refers, however, only to the characteristics of the radiation proper, while the effect of synchrotron radiation on electron motion (quantum excitation of betatron oscillations) manifests itself at much lower energies, $\varepsilon \gtrsim mc^2\,(H_0/H)^{1/4} \geqslant 500$ MeV (for $H \sim 10^4$ G) (see [2] and Section 36). Another effect also becomes important at these energies, namely, the self-polarization of the electron spin (see Section 29.5 below).

In the case where $\xi \gg 1$ the radiation is essentially of a quantum nature: the emitted photon carries off almost the entire energy of the electron, as a result of which electron recoil is considerable (see (29.68) and (29.68a)). The spectrum and radiated power of synchrotron radiation differ markedly from the respective classical charac-

teristics. Let us find the radiated power for this case. It is expedient to integrate by parts in (29.74):

$$W = W_{\mathrm{cl}} \frac{9\sqrt{3}}{8\pi} \int_0^\infty dy \left[\frac{1}{2} \frac{y^2}{(1+\xi y)^2} K_{5/3}(y) + \frac{\xi^2 y^3}{(1+\xi y)^4} K_{2/3}(y) \right], \tag{29.77}$$

where the integrand is no more the spectral distribution.

For $\xi \gg 1$ the main contribution to the integral is provided by the region with $y \ll 1$, where the well-known asymptotic representation for the functions $K_p(y)$ holds true:

$$K_p(y) \simeq 2^{p-1}\Gamma(p)\, y^{-p}. \tag{29.78}$$

Using this asymptotic representation, the integral

$$\int_0 \frac{y^p\, dy}{(1+\xi y)^q} = \xi^{-(p+1)}\Gamma(p+1)\,\Gamma(q-p-1)/\Gamma(q), \tag{29.79}$$

and (29.77), we find that for $\xi \gg 1$

$$W \simeq \frac{2^{8/3}}{9}\Gamma(2/3)\, W_{\mathrm{cl}} \xi^{-4/3} = \frac{2^4\Gamma(2/3)}{3^5}\alpha \frac{m^2c^4}{\hbar}(3\chi)^{2/3}, \tag{29.80}$$

with $\alpha = e^2/\hbar c$.

Note that by integrating the synchrotron-radiation spectrum (29.65) with respect to variable u (where the first term is integrated by parts twice) we can represent the total radiated power in the form* [13]

$$W = W_{\mathrm{cl}} \left(-\frac{3}{4\sqrt{\pi}} \right) \int_0^\infty \frac{4+5u+4u^2}{(1+u)^4} x\Phi'(x)\, dx, \quad u = \chi x^{3/2}. \tag{29.81}$$

For $\chi \ll 1$ and $\chi \gg 1$ we can now easily find formulas (29.76) and (29.80), respectively.

29.5 The Self-Polarization Effect

Let us now show that synchrotron radiation leads to preferred orientation of the electron spins in opposition to the magnetic field.

Using the angular spectral distribution for the emission rate, (29.60), we can obtain the following formula for the total proba-

* If in (29.77) we go over to the Airy functions via (17.50) and (17.41), we will arrive, of course, at (29.81).

bility of radiative transitions for $\xi \ll 1$ (to within terms of the order of ξ^2) [2]:

$$w(\zeta, \zeta') = \frac{5\sqrt{3}}{6} \frac{e^2}{\hbar c} \frac{c}{R} \frac{\varepsilon}{mc^2} \left\{ \frac{1+\zeta\zeta'}{2} \left[1 - \frac{16\sqrt{3}}{45} \xi + \frac{25}{18} \xi^2 - \frac{\zeta'}{5} \xi \left(1 - \frac{20\sqrt{3}}{9} \xi \right) \right] + \frac{1-\zeta\zeta'}{2} \frac{1}{6} \xi^2 \left(1 - \zeta' \frac{8\sqrt{3}}{15} \right) \right\}. \tag{29.82}$$

The effect we are interested in is related to the rate of transitions with spin-flip ($\zeta \to \zeta' = -\zeta$), which is given by the term in (29.82) proportional to $(1 - \zeta\zeta')/2$:

$$w(\zeta) = \frac{1}{2\tau} \left(1 + \zeta \frac{8\sqrt{3}}{15} \right), \tag{29.83}$$

where

$$\tau = \frac{8}{5\sqrt{3}} \frac{\hbar^3}{e^2 mc} \left(\frac{mc^2}{\varepsilon} \right)^2 \left(\frac{H_0}{H} \right)^3 \tag{29.84}$$

is the relaxation time (see below).

Formula (29.83) shows that the rate of transitions from the initial state with the electron spin directed along the magnetic field ($\zeta = 1$) into a final state with the electron spin in opposition to the field ($\zeta' = -1$) is considerably higher than that of reverse transitions ($\zeta = -1 \to \zeta' = +1$). Thus, as a result of photon emission, the electron spins will tend to orient themselves in opposition to the magnetic field. In the ensemble of electrons in a beam the degree of polarization is

$$P = \frac{N_\downarrow - N_\uparrow}{N_\downarrow + N_\uparrow}, \tag{29.85}$$

where $N_\uparrow$ ($N_\downarrow$) is the number of electrons with spins directed along (in opposition to) the field, and $N_\uparrow + N_\downarrow = N =$ constant.

For $N_\uparrow$ and $N_\downarrow$ we can write the following system of equations:

$$\frac{dN_\downarrow}{dt} = N_\uparrow w(1) - N_\downarrow w(-1), \quad \frac{dN_\uparrow}{dt} = N_\downarrow w(-1) - N_\uparrow w(1). \tag{29.86}$$

For the case of an initially nonpolarized beam ($P(0) = 0$), the solution to (29.86) yields the following expression for the degree of polarization:

$$P(t) = P_0 (1 - e^{-t/\tau}). \tag{29.87}$$

For $t \gg \tau$ the polarization tends to the limit

$$P_0 = \frac{8}{5\sqrt{3}} \simeq 0.924. \tag{29.88}$$

In this event approximately 96 percent of the electrons have spins that are in opposition to the magnetic field. On the other hand, positron spins will be directed preferably along the field. The degree of polarization of a positron beam is given, of course, by the same formula (29.88).

For modern storage rings, the relaxation time τ given by (29.84) proves to be of the order of one hour, while such devices operate for 24 hours or even longer. The theory of the self-polarization effect agrees well with the experimental findings (see Section 36).

30. Compton Scattering

Scattering of photons by free electrons is known as Compton scattering. In the classical theory the scattering of an electromagnetic wave by an electron leads to a scattered wave caused by photon emission of electrons moving in the field of the incident wave, with the wave frequency ω not changing in the course of scattering, since the electron oscillates with the same frequency ω. In the quantum theory this process is described as the collision of a photon of energy $\hbar\omega$ with an electron, which recoils. As a result the frequency of the scattered photon is shifted with respect to the frequency of the incident photon. Obviously, the higher the frequency of the incident radiation, the greater the shift in frequency. The first to observe this effect was the American physicist Arthur Compton (in 1922), who studied X-ray scattering by atomic electron shells.* This effect became known as the Compton effect.

30.1 The Matrix Element of the Process

According to Feynman's rules (Section 26), two second-order diagrams correspond to the scattering of a photon by an electron (Figure 30.1), which diagrams differ in the permutation of the external photon lines. The corresponding matrix element is

$$A_{fi} = (-ie\sqrt{4\pi})^2 \bar{u}' \left(\hat{e}'^* \frac{1}{\hat{p}+\hat{k}-m} \hat{e} + \hat{e} \frac{1}{\hat{p}-\hat{k}'-m} \hat{e}'^* \right) u, \quad (30.1)$$

where the first term corresponds to diagram (a) and the second, to diagram (b). Here and in what follows we use the notation $\hat{q} \equiv \gamma^\mu q_\mu$ for the "scalar" product of the 4-vector q^μ and the "4-vector" composed of the γ-matrices, $k^\mu = (\omega, \mathbf{k})$ and $k'^\mu = (\omega', \mathbf{k}')$ are the 4-momenta of the incident and scattered photons, e^μ and e'^μ are their polarization 4-vectors, and $u = u(p, \sigma)$ and $u' = u(p', \sigma')$ are the bispinor

* Since X-ray photons have high energies, the binding energy of the electrons in an atom can be ignored and the electrons considered free.

amplitudes of the initial and final electrons with momenta $p^\mu = (\varepsilon, \mathbf{p})$ and $p'^\mu = (\varepsilon', \mathbf{p}')$ and spin projections σ and σ', respectively. In (30.1) we have allowed for 4-momentum conservation at the vertices,

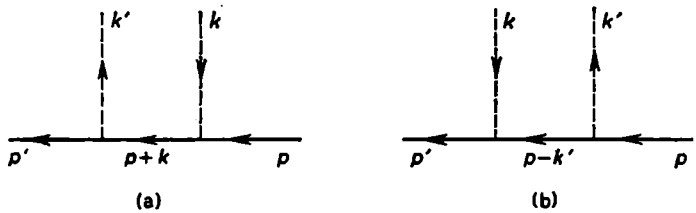

Figure 30.1

which ensures conservation of the total 4-momentum:

$$p + k = p' + k'. \tag{30.2}$$

We note also that A_{fi} is the reduced matrix element connected with the S-matrix element through (27.5).

30.2 Compton-Scattering Cross Section

Knowing the matrix element, we can find the differential cross section of the considered two-particle reaction $\gamma + e \to \gamma' + e'$ via the general formula (27.13):

$$d\sigma = \frac{1}{4(pk)} (2\pi)^4 \delta_4 (p + k - p' - k') \, |A_{fi}|^2 \frac{d^3p'}{(2\pi)^3 \, 2\varepsilon'} \frac{d^3k'}{(2\pi)^3 \, 2\omega'}, \tag{30.3}$$

where we have allowed for the fact that the invariant flux $I = ((pk)^2 - p^2k^2)^{1/2} = (pk)$, by virtue of $k^2 = 0$.

Here we will not discuss polarization effects and therefore will average over the initial polarization states and sum over the final polarization states of the particles participating in the reaction.

In accord with the results of Section 27 (see (27.19)), the square of the matrix element of the form

$$A_{fi} = \bar{u}'_\alpha Q_{\alpha\beta} u_\beta \tag{30.4}$$

averaged over the initial spin states and summed over the final spin states is given by the formula

$$\frac{1}{2} \sum_{\text{spin}} |A_{fi}|^2 = \frac{1}{2} \operatorname{Tr} \{\bar{Q} (\hat{p}' + m) \, Q (\hat{p} + m)\}. \tag{30.5}$$

The matrix $\bar{Q} = \gamma^0 Q^+ \gamma^0$ can be constructed by means of the easily verifiable formula

$$\overline{\gamma^\mu \gamma^\nu \ldots \gamma^\lambda \gamma^\rho} = \gamma^0 (\gamma^\mu \ldots \gamma^\rho)^+ \gamma^0 = \gamma^\rho \gamma^\lambda \ldots \gamma^\nu \gamma^\mu \tag{30.6}$$

and, in particular,

$$\overline{\gamma^\mu} = \gamma^\mu . \tag{30.6a}$$

In the case at hand, A_{fi} has the form (30.1), and combining (30.5) with the conservation law (30.2) yields

$$\frac{1}{2}\sum_{\text{spin}} |A_{fi}|^2 = (4\pi e^2)^2 \frac{1}{2} \operatorname{Tr}\{[\hat{e}(\hat{p}+\hat{k}-m)^{-1}\hat{e}'$$
$$+\hat{e}'(\hat{p}-\hat{k}'-m)^{-1}\hat{e}](\hat{p}+\hat{k}-\hat{k}'+m)[\hat{e}'(\hat{p}+\hat{k}-m)^{-1}\hat{e}$$
$$+\hat{e}(\hat{p}-\hat{k}'-m)^{-1}\hat{e}'](\hat{p}+m)\}. \tag{30.7}$$

For further calculations it is expedient to recall the following properties of γ-matrices (see Section 8):

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad (\gamma^k)^2 = -1, \quad (\gamma^0)^2 = 1,$$
$$\gamma^+ = -\gamma, \quad (\gamma^0)^+ = \gamma^0, \quad \operatorname{Tr}\gamma^\mu = 0, \tag{30.8}$$
$$\operatorname{Tr}\gamma^{\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_{2n+1}} = 0, \quad \operatorname{Tr}\gamma^\mu\gamma^\nu = g^{\mu\nu}.$$

This results in the following useful formulas:

$$\hat{a}\hat{b} + \hat{b}\hat{a} = 2(ab), \quad \hat{a}\hat{a} = a^2, \quad \frac{1}{4}\operatorname{Tr}\hat{a}\hat{b} = (ab),$$
$$\frac{1}{4}\operatorname{Tr}(\hat{a}\hat{b}\hat{c}\hat{d}) = (ab)(cd) - (ac)(bd) + (ad)(bc). \tag{30.9}$$

In calculating the trace (30.9) we allow for the fact that

$$e^2 = e'^2 = -1, \quad ek = e'k' = 0, \quad k^2 = k'^2 = 0. \tag{30.10}$$

It is also convenient to perform the gauge transformation

$$e \to \tilde{e} = e - \frac{(pe)}{(pk)}k, \quad e' \to \tilde{e}' = e' - \frac{(pe')}{(pk')}k', \tag{30.11}$$

which preserves the validity of (30.10) and, by virtue of

$$\tilde{e}p = 0, \ \tilde{e}'p = 0, \tag{30.12}$$

considerably simplifies calculations. In what follows we will denote, for compactness of notation, the transformed polarization vectors by e and e', as is common practice.

Allowing for (30.10) and (30.12), we get

$$\hat{e}\hat{p} = -\hat{p}\hat{e}, \ \hat{e}\hat{k} = -\hat{k}\hat{e}, \ \hat{k}\hat{k} = 0, \quad \hat{e}\hat{e} = -1,$$
$$\hat{p}\hat{p} = m^2, \ \hat{e}\hat{k}\hat{e} = \hat{k}, \ \hat{e}\hat{p}\hat{e} = \hat{p} \tag{30.13}$$

We also note the validity of the following identities:

$$\begin{aligned}(-\hat{p}+m)(\hat{p}+\hat{k}-m)^{-1} &= \frac{1}{2(pk)}(-\hat{p}+m)(\hat{p}+\hat{k}+m)\\ &= (-\hat{p}+m)\frac{\hat{k}}{2pk},\\ (\hat{p}+\hat{k}-m)^{-1}(-\hat{p}+m) &= \frac{\hat{k}}{2(pk)}(-\hat{p}+m).\end{aligned} \tag{30.14}$$

Using the second identity and the fact that $\hat{e}(\hat{p}+m) = (-\hat{p}+m)\hat{e}$ (see (30.13)), we can shift $\hat{p}+m$ on the right-hand side of (30.7) to the left as far as the factor $\hat{p}+\hat{k}-\hat{k}'+m$ and then return it to its former place via the identity:

$$(\hat{p}+m)\hat{e}'\hat{k}\hat{e} = \hat{e}'\hat{k}\hat{e}(\hat{p}+m) - 2(kp)\hat{e}'\hat{e}. \tag{30.15}$$

After this, using the well-known property of trace, Tr $(AB \ldots X) =$ Tr $(XAB \ldots)$, we take $\hat{p}+m$ to the left-hand side and move it to the right as far as the factor $\hat{p}+\hat{k}-\hat{k}'+m$ via the first identity in (30.14) and then return it back using a relation similar to (30.15). As a result, (30.17) takes the form

$$\begin{aligned}\frac{1}{2}\sum_{\text{spin}}|A_{fi}|^2 &= (4\pi e^2)^2\frac{1}{2}\operatorname{Tr}\{(\hat{p}+m)\hat{K}(\hat{p}+\hat{k}-\hat{k}'+m)\hat{K}'\}\\ &= (4\pi e^2)^2[C_1+C_2],\\ \hat{K} &= \frac{\hat{e}\hat{k}\hat{e}'}{2(kp)}+\frac{\hat{e}'\hat{k}'\hat{e}}{2(k'p)},\quad \hat{K}' = \hat{K}(\hat{e}\rightleftarrows\hat{e}'),\end{aligned} \tag{30.16}$$

where $\hat{K}'$ is obtained from $\hat{K}$ through the noted substitution; here we have introduced the notation

$$\begin{aligned}C_1 &= \frac{1}{2}\operatorname{Tr}\{(\hat{p}+m)\hat{K}(\hat{p}+m)\hat{K}'\},\\ C_2 &= \frac{1}{2}\operatorname{Tr}\{(\hat{p}+m)\hat{K}(\hat{k}-\hat{k}')\hat{K}'\}.\end{aligned} \tag{30.17}$$

The partitioning of (30.16) into two terms has a clear physical meaning. The term C_1 corresponds to the contribution to the scattering cross section that does not allow for electron recoil ($p = p'$), while C_2 allows for recoil. In the classical limit, obviously, we are left only with C_1 ($C_2 \to 0$ as $k' \to k$).

To calculate C_1, we shift the right $\hat{p} + m$ to the left:

$$C_1 = \frac{1}{2}\operatorname{Tr}\{(\hat{p}+m)[(m-\hat{p})\hat{K}-(\hat{e}\hat{e}'+\hat{e}'\hat{e})]\hat{K}\}$$

$$= -2(ee')\frac{1}{2}\operatorname{Tr}\{(\hat{p}+m)\hat{K}'\} = -(ee')\operatorname{Tr}(\hat{p}\hat{K}') = 4(ee')^2, \quad (30.18)$$

where we have used the relations $(\hat{p} + m)(\hat{p} - m) = p^2 - m^2 = 0$, $\operatorname{Tr}\hat{K}' = 0$, (30.9), and (30.10).

Next we calculate C_2 by writing it out explicitly:

$$C_2 = \frac{1}{2}\operatorname{Tr}\left\{\hat{p}\left[\frac{\hat{e}\hat{k}\hat{e}'}{2kp}+\frac{\hat{e}'\hat{k}'\hat{e}}{2k'p}\right](\hat{k}-\hat{k}')\left[\frac{\hat{e}'\hat{k}\hat{e}}{2kp}+\frac{\hat{e}\hat{k}'\hat{e}'}{2k'p}\right]\right\}.$$

Here we have discarded terms proportional to m since they contain an odd number of γ-matrices and, therefore, are traceless (see (30.8)). The terms whose products provide a nonzero contribution to the trace are connected by lines (all other combinations yield zero). Thus,

$$C_2 = \frac{1}{2}\operatorname{Tr}\left\{\frac{\hat{p}\hat{e}'\hat{k}'(\hat{e}\hat{k}\hat{e})\hat{k}'\hat{e}'}{(2k'p)^2}-\frac{\hat{p}\hat{e}\hat{k}(\hat{e}'\hat{k}'\hat{e}')\hat{k}\hat{e}}{(2kp)^2}\right\}$$

$$= \frac{1}{2}\frac{1}{4}\operatorname{Tr}\left\{\frac{1}{(k'p)^2}\hat{p}\hat{k}'\hat{k}\hat{k}'-\frac{1}{(kp)^2}\hat{p}\hat{k}\hat{k}'\hat{k}\right\}$$

$$= \frac{kk'}{pk'}-\frac{kk'}{pk} = \frac{pk}{pk'}+\frac{pk'}{pk}-2. \quad (30.19)$$

Here we have allowed for (30.13) and

$$kk' = p(k-k'), \quad (30.20)$$

which follows from the conservation law (30.2).

Substituting (30.18) and (30.19) into (30.16) and the result into (30.3), we can write the formula for the cross section of scattering of a photon by a nonpolarized electron:

$$\sigma = \int d\sigma = \frac{e^4}{(kp)}\int \delta_4(p+k-p'-k')\frac{d^3k'\,d^3p'}{4\omega'\varepsilon'}C(\tilde{e},\tilde{e}'), \quad (30.21)$$

$$C(\tilde{e},\tilde{e}') = \frac{pk}{pk'}+\frac{pk'}{pk}-2+4(\tilde{e}\tilde{e}')^2,$$

where we have returned to the initial notation for the transformed polarization vectors (30.11).

Noting that

$$\int \frac{d^3k'}{2\omega'}\frac{d^3p'}{2\varepsilon'} = \int d^4k'\,d^4p'\delta(k'^2)\delta(p'^2-m^2),$$

we can express the cross section (30.21) in invariant form

$$\sigma = \frac{\alpha^2}{kp} \int d^4k' \delta(k'^2)\, \delta((p+k-k')^2 - m^2)\, C(\tilde{e}, \tilde{e}'), \quad (30.21a)$$

where $\alpha = e^2$ $(=e^2/\hbar c)$.

Now let us average the result over the initial photon polarization states and sum over the final polarization states:

$$C \equiv \frac{1}{2} \sum_{\text{pol}} C(\tilde{e}, \tilde{e}') = 2\left(\frac{pk}{pk'} + \frac{pk'}{pk} - 2\right) + 2 \sum_{\text{pol}} (\tilde{e}\tilde{e}')^2. \quad (30.22)$$

Expressing vectors $\tilde{e}$ and $\tilde{e}'$ in terms of the initial vectors e and e' according to (30.11) and allowing for the summation formula $\sum_{\text{pol}} e_\mu e'_\nu = -g_{\mu\nu}$ (see (27.27)), we find that

$$\begin{aligned} \sum_{\text{pol}} (\tilde{e}\tilde{e}')^2 &= \sum_{\text{pol}} \left(e^\mu - \frac{pe}{pk} k^\mu\right)\left(e^\nu - \frac{pe}{pk} k^\nu\right) \\ &\quad \times \left(e'_\mu - \frac{pe'}{pk'} k'_\mu\right)\left(e'_\nu - \frac{pe'}{pk'} k'_\nu\right) \\ &= 2\left[1 + \frac{m^4}{2} \frac{(kk')^2}{(pk)^2 (pk')^2} - m^2 \frac{kk'}{(pk)(pk')}\right]. \end{aligned} \quad (30.23)$$

Combining (30.23), (30.22), and (30.21a), we arrive at the following formula for the scattering cross section for nonpolarized particles:

$$\sigma = \frac{\alpha^2}{kp} \int \frac{d^3k'}{2\omega'} \delta((p+k-k')^2 - m^2)\, C, \quad (30.24)$$

$$C = 2\left[\frac{pk}{pk'} + \frac{pk'}{pk} + m^4 \frac{(kk')^2}{(pk)^2 (pk')^2} - 2m^2 \frac{(kk')}{(pk)(pk')}\right].$$

Let us introduce the invariant spectral variable u and the parameter $\varkappa$ thus:

$$u = \frac{kk'}{kp'}, \quad \varkappa = \frac{2kp}{m^2}. \quad (30.25)$$

Combining this with $kp' = kp - kk'$ (see (30.2)) and (30.20), we get

$$u = \frac{kk'}{kp - kk'}, \quad kk' = kp \frac{u}{1+u}, \quad pk' = \frac{kp}{1+u}. \quad (30.26)$$

Substituting (30.26) into the expression for C in (30.24) yields

$$C \equiv C(\varkappa, u) = 2 + \frac{u^2}{1+u} - 4\frac{u}{\varkappa}\left(1 - \frac{u}{\varkappa}\right). \quad (30.27)$$

Now let us write the phase volume $d^3k'/2\omega'$ in terms of invariant variables. It is convenient to perform all calculations in the center-of-mass system, in which

$$\mathbf{p} + \mathbf{k} = \mathbf{p}' + \mathbf{k}' = 0. \tag{30.28}$$

The presence of the delta function in (30.24) implies the relationship

$$x \equiv (p + k - k')^2 - m^2 = 2(pk - pk' - kk') = 0. \tag{30.29}$$

By virtue of (30.28) we have (in the center-of-mass system)

$$\begin{aligned} kp &= \omega\varepsilon - \mathbf{k}\cdot\mathbf{p} = \omega\varepsilon + \omega^2 = \omega(\varepsilon + \omega), \\ k'(p + k) &= \omega'(\varepsilon + \omega), \end{aligned} \tag{30.30}$$

and (30.29) is reduced to

$$x = 2(\varepsilon + \omega)(\omega - \omega') = 0, \tag{30.29a}$$

that is, in the center-of-mass system the frequency does not change, $\omega' = \omega$.

Allowing for (30.29a), we get

$$d\Gamma \equiv \frac{d^3k'}{2\omega'}\delta(x) = \frac{1}{2\omega'}\frac{1}{2(\varepsilon+\omega)}\omega'^2 d\omega'\, d\cos\theta\, d\varphi\delta(\omega - \omega'), \tag{30.31}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$, φ is the azimuthal angle of vector $\mathbf{k}'$ in a plane perpendicular to $\mathbf{k}$.

The variable u in the center-of-mass system assumes the form

$$0 \leqslant u = \frac{kp}{kp - \omega^2(1 - \cos\theta)} - 1 \leqslant \varkappa, \tag{30.32}$$

where we have allowed for (30.26) and the fact that $kk' = \omega^2(1 - \cos\theta)$ and where the lower (upper) limit of variation of u corresponds to $\theta = 0$ ($\theta = \pi$). Whence

$$d\cos\theta = \frac{kp}{\omega^2}\frac{du}{(1+u)^2}, \tag{30.33}$$

and the formula for the phase volume, (30.31), assumes the form

$$d\Gamma = \frac{1}{4}d\varphi\, d\omega'\delta(\omega - \omega')\frac{du}{(1+u)^2}. \tag{30.31a}$$

We substitute (30.31a) into (30.24), integrate with respect to φ, which yields a factor of 2π, and then integrate with respect to ω' via the delta function. Combining the result with (30.27), we arrive at the final form for the Compton-scattering cross section:

$$\sigma = \frac{2\pi r_0^2}{\varkappa}\int_0^{\varkappa}\frac{du}{(1+u)^2}\left[2 + \frac{u^2}{1+u} - 4\frac{u}{\varkappa}\left(1 - \frac{u}{\varkappa}\right)\right], \tag{30.34}$$

where $r_0 = \alpha/m$ $(=e^2/mc^2)$ is the classical electron radius, and the integrand determines the differential cross sections (in the variable u).

Integrating with respect to u, we obtain

$$\sigma = 2\pi r_0^2 \left[\left(\frac{1}{\varkappa} - \frac{4}{\varkappa^2} - \frac{8}{\varkappa^3} \right) \ln (1+\varkappa) + \frac{1+\varkappa/2}{(1+\varkappa)^2} + \frac{8}{\varkappa^2} \right]. \quad (30.35)$$

The asymptotic behavior of the cross section is as follows:

$$\sigma = \begin{cases} \frac{8\pi r_0^2}{3}(1-\varkappa) & \text{if} \quad \varkappa \ll 1, \\ \frac{2\pi r_0^2}{\varkappa}\left(\ln \varkappa + \frac{1}{2}\right) & \text{if} \quad \varkappa \gg 1. \end{cases} \quad (30.35a)$$

For $\varkappa = 0$ this formula yields the classical Thomson formula $\sigma_T = \sigma\,(\varkappa = 0) = (8\pi/3)\, r_0^2$. As $\varkappa$ tends to infinity, the cross section diminishes according to the law $\sigma\,(\varkappa) \propto (\ln \varkappa)/\varkappa$.

30.3 Scattering in the Laboratory System

Let us consider the scattering process in the laboratory system. In this system the initial electron is at rest ($\mathbf{p} = 0$) and parameter $\varkappa$ given by (30.25) is (in ordinary units)

$$\varkappa = 2\hbar\omega/mc^2. \quad (30.36)$$

The condition $\varkappa \ll 1$ means, therefore, that the quantum effects are small (electron recoil in scattering is small). For $\varkappa \gg 1$ the scattering is essentially quantum. From (30.20) we find the relation between the scattered photon frequency ω' and the scattering angle θ (in the laboratory system):

$$\omega\omega'\,(1 - \cos\theta) = m\,(\omega - \omega'),$$

or

$$\omega' = \frac{\omega}{1 + \frac{\hbar\omega}{mc^2}(1 - \cos\theta)}. \quad (30.37)$$

The shift in wavelength $\lambda = 2\pi c/\omega$ caused by scattering is

$$\Delta\lambda = \lambda' - \lambda = 2\pi\lambda_e\,(1 - \cos\theta), \quad (30.37a)$$

where $\lambda_e = \hbar/mc \simeq 3.86 \times 10^{-11}$ cm is the Compton wavelength of the electron, and formula (30.37a) is known as the Compton formula. According to (30.37), in the laboratory system the frequency of the scattered photon lowers (or the wavelength increases).

Bearing in mind that, according to (30.26),

$$\omega' = \frac{\omega}{1+u}, \quad (30.38)$$

we can express the differential cross section (see (30.34)) in terms of the energy of the scattered photon, ω':

$$\frac{d\sigma}{d\omega'} = \pi r_0^2 \frac{m}{\omega^2}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \left(\frac{m}{\omega'} - \frac{m}{\omega}\right)^2 - 2m\left(\frac{1}{\omega'} - \frac{1}{\omega}\right)\right], \quad (30.39)$$

$$\frac{\omega}{1+2\omega/m} \leqslant \omega' \leqslant \omega.$$

By virtue of (30.37), the frequency ω' is uniquely related to the scattering angle θ. Introducing the solid angle element $d\Omega = 2\pi \sin\theta\, d\theta$, we can write the angular distribution of scattered photons thus:

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right). \quad (30.40)$$

This is known as the Klein-Nishina-Tamm formula. In the classical limit, where $\hbar\omega \ll mc^2$, we can put $\omega' \simeq \omega$, and (30.40) becomes the well-known Thomson formula

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 (1 + \cos^2\theta), \quad (30.41)$$

whose integration again yields $\sigma = \sigma_T = (8\pi/3)\, r_0^2$, the limit of (30.37a) at $\varkappa = 0$.

30.4 Inverse Compton Effect

Let us investigate the special case of the scattering of soft photons ($\omega \ll m$) by relativistic electrons ($\varepsilon \gg m$, $\varepsilon' \gg m$), with the photons initially flying towards the electrons ($\mathbf{k} \uparrow\downarrow \mathbf{p}$). As a result of head-on collisions, a beam of energetic photons is produced, with the photons in the beam moving along the initial electron momentum $\mathbf{p}$.

In the laboratory system we have the following formula for the spectral variable:

$$u = \frac{kk'}{kp'} = \frac{\omega\omega'(1-\cos\theta)}{\omega(\varepsilon' - |\mathbf{p}'|\cos\psi)} \simeq \frac{\omega'}{\varepsilon'},$$

since $\cos\theta \simeq -1$, $\cos\psi \simeq -1$ (with $\psi = \widehat{(\mathbf{p}', \mathbf{k})}$), and $|\mathbf{p}'| \simeq \varepsilon'$. Moreover, since $\varepsilon' = \varepsilon + \omega - \omega' \simeq \varepsilon - \omega'$, we have

$$0 \leqslant u = \frac{\omega'}{\varepsilon - \omega'} \leqslant \varkappa = \frac{2kp}{m^2}. \quad (30.42)$$

The parameter is given by the formula

$$\varkappa = \frac{2\omega(\varepsilon + |\mathbf{p}|)}{m^2} \simeq \frac{4\omega\varepsilon}{m^2}. \quad (30.43)$$

Hence, the maximum frequency of the scattered photons,

$$\omega'_m = \frac{\varkappa}{1+\varkappa}\varepsilon, \tag{30.44}$$

is close, for $\varkappa \gg 1$ ($\varepsilon \gg m^2/\omega$), to the energy of the initial electrons.

Substituting u from (30.42) into (30.38), we get the formula for the spectrum of scattered photons:

$$\frac{d\sigma}{d\omega'} = \frac{2\pi r_0^2}{\varkappa\varepsilon}\left[1 + \frac{\omega'^2}{\varepsilon(\varepsilon-\omega')} + \left(\frac{2\omega'}{\varkappa(\varepsilon-\omega')} - 1\right)^2\right]. \tag{30.45}$$

The spectrum increases monotonically to its maximum at $\omega' = \omega'_m$, with the relative width of the maximum being (for $\varkappa \gg 1$, or $\omega'_m \simeq \varepsilon$)

$$\frac{\Delta\omega'}{\omega'_m} = \frac{\omega'_m - \omega'_{1/2}}{\omega'_m} \simeq \varkappa^{-1} \ll 1, \tag{30.46}$$

where $\omega'_{1/2}$ is the frequency corresponding to the half-height of $d\sigma/d\omega'$. Thus, for large $\varkappa$'s the spectrum has a sharp maximum at $\omega' \simeq \varepsilon$ whose width rapidly decreases as $\varkappa$ grows (and the spectrum approaches a monochromatic one).

The scattering of soft photons by relativistic electrons accompanied by emission of energetic photons is known as the inverse Compton effect (in contrast to the process considered in Section 30.3 in which photons were scattered by electrons at rest, which led to a decrease in the photon frequency in accord with (30.37)). This effect is important in astrophysical applications. For instance, such a mechanism has been extensively considered as a source for the production of X-rays in cosmic objects and for the production of X-ray component in the galactic background radiation.

The inverse Compton effect is also used in laboratory studies as a source of high-energy gamma rays. For instance, if a beam of 6-GeV electrons (corresponding, say, to the electron beam in the ARUS synchrotron in Yerevan, the capital of the Armenian Soviet Republic) is irradiated by a ruby-laser beam of photons with $\omega = 1.78$ eV, then $\varkappa \simeq 0.17$ and the maximum energy of the scattered photons (see (30.44)) is $\omega'_m \simeq \varkappa\varepsilon \simeq 1$ GeV, that is, the wavelength $\lambda_m = c/\omega'_m \simeq 2 \times 10^{-14}$ cm is shorter by a factor of 10^3 than the electron Compton wavelength and smaller by a factor of almost 10 than the radius of nuclear forces (of the order of 10^{-13} cm).

31. Crossing Symmetry

The considered effect of Compton scattering constitutes a particular case of a reaction represented by a diagram with two external photon and electron lines (Figure 31.1). According to the general theory (see Sections 25 and 26), an incoming electron line has corresponding to

it either an initial electron with momentum p (amplitude $u(p)$) or a final positron with the same momentum (amplitude $u(-p)$), while an outgoing electron line has corresponding to it either a final electron with momentum p' (amplitude $\bar{u}(p')$) or an initial positron with the same momentum (amplitude $\bar{u}(-p')$). Thus, apart from describing the process when a photon is scattered by an electron, the diagram also describes the following processes: the scattering of a photon by a positron, two-photon annihilation of an electron-positron pair, and creation of an electron-positron pair from two photons.* All four processes, precisely, the matrix elements for these processes, are connected, and this property became known as crossing symmetry, relating the matrix elements for apparently dissimilar physical phenomena (see below). The same relationship exists between matrix elements of any reactions described by topologically equivalent diagrams (i.e. diagrams obtained through interchanging the vertices and the meaning of the incoming and outgoing external lines).

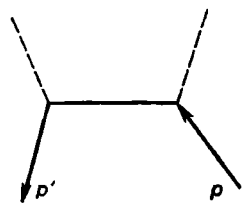

Figure 31.1

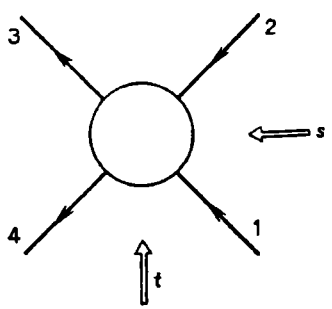

Figure 31.2

Let us consider the reaction in which two particles in the initial state transform into two particles in the final state:

$$1 + 2 \to 3 + 4, \qquad (31.1)$$

where the number labels the particle. To this reaction we assign the general diagram depicted in Figure 31.2, where the circle stands for the diagram's inner part, which is of no interest to us now, and all the lines are depicted in the same manner (the type of particles and the interaction between them are not specified).

By 4-momentum conservation,

$$p_1 + p_2 = p_3 + p_4. \qquad (31.2)$$

To simplify matters, we assume that the particles have the same mass (the general conclusions that we will arrive at shortly do not depend on this assumption, which is not satisfied usually in the

* Formally this general diagram also describes processes with one particle in the initial (final) state and three particles in the final (initial) state. However, the matrix elements of such processes (in electrodynamics) vanish by virtue of 4-momentum conservation (the argument in the delta function that expresses this conservation law is always nonzero for such reactions).

case of electrodynamics), that is,

$$p_i^2 = m^2 \quad (i = 1, 2, 3, 4). \tag{31.3}$$

Let us introduce two invariant variables:

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_4)^2. \tag{31.4}$$

Of course, we could have built other invariant combinations of 4-momenta p_i, but by virtue of (31.2) such combinations will be functions of s and t. Hence, only two independent invariant variables characterize reaction (31.1), with s corresponding to the initial state and t to the final. In the center-of-mass system involving particles 1 and 2, we have

$$\begin{gathered} \mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon, \\ \mathbf{p}_3 = -\mathbf{p}_4 = -\mathbf{p}', \quad \varepsilon_3 = \varepsilon_4 = \varepsilon, \\ |\mathbf{p}| = |\mathbf{p}'|, \end{gathered} \tag{31.5}$$

where we have allowed for (31.3). If we denote the angle between $\mathbf{p}$ and $\mathbf{p}'$ by θ, the values of variables s and t in this system are

$$\begin{gathered} s = 4\varepsilon^2 > 4m^2, \\ t = 2m^2 - 2\varepsilon^2 + 2\mathbf{p}\cdot\mathbf{p}' = 2(m^2 - \varepsilon^2) + 2|\mathbf{p}||\mathbf{p}'|\cos\theta \\ = -\frac{1}{2}(s - 4m^2)(1 - \cos\theta) < 0. \end{gathered} \tag{31.6}$$

Besides reaction (31.1), the following reaction is possible:

$$1 + \bar{4} \to 3 + \bar{2}, \tag{31.7}$$

where the bar denotes the respective antiparticle. This reaction obeys the following 4-momentum conservation law:

$$p_1 + p_{\bar{4}} = p_3 + p_{\bar{2}}. \tag{31.8}$$

If $p_{\bar{4}} = -p_4$ and $p_{\bar{2}} = -p_2$, then (31.8) becomes (31.2). In this case the variables (31.4) assume the form

$$s = (p_1 + p_2)^2 = (p_1 - p_{\bar{2}})^2, \quad t = (p_1 - p_4)^2 = (p_1 + p_{\bar{4}})^2, \tag{31.9}$$

and in the center-of-mass system of particles 1 and $\bar{4}$ these variables are

$$t = 4\varepsilon^2 > 4m^2, \quad s = -\frac{1}{2}(t - 4m^2)(1 - \cos\theta) < 0. \tag{31.10}$$

Reaction (31.1) is known as the s-channel of the general reaction described by the block diagram in Figure 31.2 with four external lines. In this channel $s > 4m^2$ and $t < 0$. Correspondingly, reaction (31.8) is known as the t-channel, in which $t > 4m^2$ and $s < 0$.

It can be verified by direct calculation of the amplitudes A_{fi} (matrix elements) using concrete Feynman diagrams that the reactions in the s- and the t-channel are determined by one and the same amplitude $A(s, t)$ ($A_{fi} \propto A$), which is a unique analytic function of variables s and t in both channels, and differ only in the ranges of s and t in the two channels. The channels are said to be crossed, and the property of amplitude A is known as crossing symmetry.

To transfer from the s-channel to the t-channel (they are designated in Figure 31.2 by double arrows), we must, obviously, perform the following transformations in the matrix element A_{fi} (compare (31.1) with (31.8)):

$$(p_{\bar{a}})_i = -(p_a)_f, \quad (p_{\bar{2}})_f = -(p_2)_i, \tag{31.11}$$

where subscripts i and f on the left (right) label the initial and final states in the t-channel (s-channel).

Crossing symmetry makes it possible, knowing the cross section in one channel, to easily obtain the cross section of the reaction in another channel (the matrix element can be written immediately, and what remains to be done is to transform the phase volume of the reaction products).

32. Two-Photon Creation and Annihilation of Electron-Positron Pairs

Both processes are described by a single block diagram (Figure 32.1). Their cross sections can be obtained by employing the Compton-scattering cross section (Section 30) and crossing symmetry (Section 31).

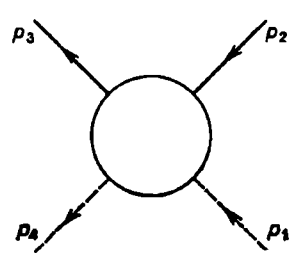

Figure 32.1

For these reactions we denote the 4-momenta of the electron and positron by p_- and p_+, respectively, while the 4-momenta of the photons are denoted by k_1 and k_2. For Compton scattering, as noted earlier, p and k (p' and k') are the 4-momenta of the initial (final) electron

and photon, and in Figure 32.1 the momenta p_i have the following meaning:

$$p_1 = k, \quad p_2 = p, \quad p_3 = p', \, p_4 = k'. \tag{32.1}$$

32.1 Two-Photon Pair Creation ($\gamma\gamma \to e^+e^-$)

For this process we have (see Figure 32.1)

$$p_1 = k_1, \quad -p_4 = k_2, \quad -p_2 = p_+, \quad p_3 = p_-, \tag{32.2}$$

and comparison with (32.1) shows that the amplitude of this process is obtained from the Compton-scattering amplitude if we put

$$k = k_1, \, k' = -k_2, \quad p = -p_+, \, p' = p_-. \tag{32.3}$$

Since we are not interested in polarization effects, let us average over the initial polarizations and sum over the final polarizations the square of the pair-creation amplitude. The result is readily obtained if in C given by (30.24) we introduce (32.3):

$$\frac{1}{4} \sum_{\substack{\text{spin } e^+e^- \\ \text{pol } \gamma\gamma}} |A_{fi}|^2 = -2\left[-\frac{(p_+k_1)}{(p_+k_2)} - \frac{(p_+k_2)}{(p_+k_1)} + m^4 \frac{(k_1k_2)^2}{(k_1p_+)^2 (k_2p_+)^2} \right.$$
$$\left. -2m^2 \frac{(k_1k_2)}{(p_+k_1)(p_+k_2)} \right], \tag{32.4}$$

where the overall minus appears because summation over the spin states of the positron is obtained by replacing $\hat{p} + m$ with $\hat{p}_+ - m = -(-\hat{p}_+ + m)$ in (30.7), that is, in addition to (32.3) we must introduce the factor -1.

Let us now introduce the invariant variable u and parameter $\varkappa_1$ via the following formulas (not to be confused with (30.25)):

$$u = \frac{(k_1k_2)^2}{4(k_1p_+)(k_2p_+)}, \quad \varkappa_1 = \frac{(k_1k_2)}{2m^2}. \tag{32.5}$$

Expressing (32.4) in terms of u and $\varkappa_1$, we get

$$\frac{1}{4} \sum |A_{fi}|^2 = 4\left[2u - 1 - 2\left(\frac{u}{\varkappa_1}\right)^2 + 2\frac{u}{\varkappa_1} \right] \equiv 8P(u, \varkappa_1). \tag{32.6}$$

In the center-of-mass system of the photons we have

$$\mathbf{k}_1 + \mathbf{k}_2 = 0, \quad \omega_1 = \omega_2 = \varepsilon_+ = \varepsilon_- = \omega,$$
$$\varkappa_1 = \left(\frac{\omega}{m}\right)^2, \quad u = \frac{\omega^2}{\omega^2 - |\mathbf{p}_+|^2 \cos^2\theta}, \tag{32.7}$$

with $\theta = (\widehat{\mathbf{p}_+, \mathbf{k}_1})$. This leads to the following limits for u:

$$1 \leqslant u \leqslant \varkappa_1, \tag{32.8}$$

with the interval being traversed twice as θ is varied from 0 to π.

In the case of a head-on collision of the photons in the laboratory system ($\mathbf{k}_1 \uparrow\downarrow \mathbf{k}_2$, $\omega_1 \neq \omega_2$), $k_1k_2 = 2\omega_1\omega_2$ and the condition

$$\varkappa_1 > 1 \tag{32.9}$$

for the reaction to take place (see (32.8)) takes the form

$$\omega_1\omega_2 > m^2. \tag{32.9a}$$

Adding this inequality to the obvious inequality $(\omega_1 - \omega_2)^2/4 \geqslant 0$, we arrive at the equivalent condition

$$\omega_1 + \omega_2 > 2m, \tag{32.9b}$$

which, of course, immediately follows from energy conservation.

The cross section of the process can be found via the general rules (see Section 27) by allowing for (32.6) and the expression for the invariant flux $I = k_1k_2$:

$$\sigma = \frac{8\alpha^2}{(k_1k_2)} \int \frac{d^3p_+}{2\varepsilon_+} \frac{d^3p_-}{2\varepsilon_-} \delta_4 (k_1 + k_2 - p_+ - p_-) P(u, \varkappa_1)$$

$$= \frac{8\alpha^2}{k_1k_2} \int \frac{d^3p_+}{2\varepsilon_+} \delta((p_+ - k_1 - k_2)^2 - m^2) P(u, \varkappa_1). \tag{32.10}$$

Here we have used the same transformations as we did in deriving (30.23a).

Let us now express the phase volume in terms of the invariant variable u. In the center-of-mass system (see (32.7)), the argument of the delta function in (32.10) becomes $x \equiv (p_+ - k_1 - k_2)^2 - m^2 = 4\omega(\varepsilon_+ - \omega)$, and

$$\delta(x) = \frac{1}{4\omega} \delta(\varepsilon_+ - \omega),$$

while the phase volume element is transformed thus:

$$d^3p_+ = d\cos\theta\, d\varphi\, |\mathbf{p}_+|^2 d|\mathbf{p}_+|, \quad |\mathbf{p}_+| d|\mathbf{p}_+| = \varepsilon_+ d\varepsilon_+.$$

The integral with respect to the azimuthal angle φ yields 2π, the integral with respect to ε_+ is easily evaluated thanks to the presence of a delta function, and the integral with respect to $\cos\theta$ can be reduced to an integral with respect to u via (32.7), with a general factor of 2 introduced in accord with the remark made in connection with (32.8). As a result we arrive at the cross section of the $\gamma\gamma \to e^+e^-$ process in invariant form:

$$\sigma_{\text{creat}} = \frac{\pi r_0^2}{\varkappa_1} \int_1^{\varkappa_1} \frac{du}{u\sqrt{u(u-1)}} \left[u - \frac{1}{2} + \frac{u}{\varkappa_1} - \left(\frac{u}{\varkappa_1}\right)^2\right], \tag{32.11}$$

where the integrand gives the spectral distribution, $d\sigma/du$, in variable u given in (32.5). By a change of variable, $u = \cosh^2 t$, the integral in (32.11) can easily be evaluated. As a result we arrive at an expression for the total cross section of electron-positron pair creation in a form known as the Breit-Wheeler formula:

$$\sigma_{\text{creat}} = \frac{\pi r_0^2}{2}(1-v^2)\left[(3-v^4)\ln\frac{1+v}{1-v} - 2v(2-v^2)\right], \quad (32.12)$$

with v related to parameter $\varkappa_1$ thus:

$$\varkappa_1 = \frac{1}{1-v^2}, \qquad v = \sqrt{1-\frac{1}{\varkappa_1}}. \quad (32.13)$$

If we now go over to the center-of-mass system (see (32.7)), we immediately see that v is the speed of a component in the pair in this system:* $\varepsilon_+ = \varepsilon_- = \omega = m(1-v^2)^{-1/2}$.

32.2 Two-Photon Pair Annihilation

This process is the inverse of pair creation, whereby the amplitudes of the two processes are the same (we denote them by P). Let us compare the annihilation differential cross section $d\sigma_{\text{ann}}$ and the creation differential cross section $d\sigma_{\text{creat}}$. Obviously,

$$d\sigma_{\text{ann}} = \frac{P}{2I_{\text{ann}}}\delta_4(p_+ + p_- - k_1 - k_2)\frac{d^3k_1\,d^3k_2}{\omega_1\omega_2},$$
$$d\sigma_{\text{creat}} = \frac{P}{I_{\text{creat}}}\delta_4(p_+ + p_- - k_1 - k_2)\frac{d^3p_+\,d^3p_-}{\varepsilon_+\varepsilon_-}, \quad (32.14)$$

where the factor 1/2 in the annihilation cross section is due to the fact that the photons are identical particles. Let us evaluate the integrals with respect to $\mathbf{k}_2$ and $\mathbf{p}_-$ via the delta function $\delta(\mathbf{p}_+ + \mathbf{p}_- - \mathbf{k}_1 - \mathbf{k}_2)$ and the integrals with respect to ω_1 and $|\mathbf{p}_+|$ (note that $d|\mathbf{p}_+| = \varepsilon_+ d\varepsilon_+/|\mathbf{p}_+|$) via the delta function $\delta(\varepsilon_+ + \varepsilon_- - \omega_1 - \omega_2)$. The result is

$$\frac{d\sigma_{\text{ann}}}{d\sigma_{\text{creat}}} = \frac{I_{\text{creat}}}{2I_{\text{ann}}}\frac{\omega_1\varepsilon_-}{\omega_2|\mathbf{p}_+|}. \quad (32.15)$$

Going over to the center-of-mass system (see (32.7)), we get $I_{\text{creat}} = k_1k_2 = 2\omega^2$ and $I_{\text{ann}} = 2\omega|\mathbf{p}|$ ($\varepsilon_+ = \varepsilon_- = \omega$), and (32.15) assumes the form

$$\frac{d\sigma_{\text{ann}}}{d\sigma_{\text{creat}}} = \frac{1}{2}\left(\frac{\omega}{|\mathbf{p}|}\right)^2 = \frac{1}{2v^2}, \quad (32.15a)$$

* Note that formula (32.12) becomes invalid when $v \lesssim \alpha$, since in this event we must allow for the Coulomb interaction between the electron and the positron, which may lead to formation of a bound system, positronium [13].

where v is the speed of a component in the electron-positron pair in the center-of-mass system, that is, the invariant parameter specified in (32.13).

We note in passing that it is convenient to represent parameter $\varkappa_1$ given in (32.5) in the form

$$\varkappa_1 = \frac{(p_+ + p_-)^2}{4m^2} = \frac{(p_+ p_-)}{2m^2} + \frac{1}{2}, \tag{32.16}$$

which allows for the conservation laws $p_+ + p_- = k_1 + k_2$ and $k_1^2 = k_2^2 = 0$. The annihilation cross section can be found by combining (32.15a) and (32.12):

$$\sigma_{\text{ann}} = \frac{\sigma_{\text{creat}}}{2v^2} = \pi r_0^2 \frac{1-v^2}{4v^2}\left[(3-v^4)\ln\frac{1+v}{1-v} - 2v(2-v^2)\right]. \tag{32.17}$$

Let us consider the limiting cases for the general formula (32.17). In the nonrelativistic limit ($v \ll 1$) we get*

$$\sigma_{\text{ann}} = \frac{\pi r_0^2}{2v}, \tag{32.18}$$

with $2v = v_+$ being the speed of the positron in the laboratory system ($\mathbf{p}_- = 0$, $\mathbf{v}_- = 0$). In the ultrarelativistic limit ($1 - v^2 \ll 1$) we find that

$$\sigma_{\text{ann}} = \frac{1}{2}\pi r_0^2 (1-v^2)\left(\ln\frac{2}{1-v} - 1\right) = \frac{\pi r_0^2}{2\gamma^2}(\ln 4\gamma^2 - 1), \tag{32.19}$$

where $\gamma = (1 - v^2)^{-1/2} = \varepsilon/m$, and $\varepsilon = \varepsilon_+ = \varepsilon_-$ is the energy of the pair components in the center-of-mass system. In the laboratory system ($\mathbf{p}_- = 0$, $\varepsilon_+ \gg m$) we have

$$\sigma_{\text{ann}} = \pi r_0^2 \frac{m}{\varepsilon_+}\left[\ln\frac{2\varepsilon_+}{m} - 1\right], \tag{32.20}$$

where we have used (32.13) and (32.16).

33. The Method of Equivalent Photons

Let us consider an approximate method for calculating the cross sections of processes in which a particle with 4-momentum p and mass m collides with a charged particle with 4-momentum Q and mass M. As a result of such a collision accompanied by an exchange of a vir-

* The divergence of the cross section at $v = 0$ is fictitious, since formula (32.17) cannot be applied for $v \lesssim \alpha$ (see the remark concerning formula (32.12)).

tual photon with 4-momentum k, the particle with mass m transforms into an ensemble of particles with total 4-momentum p'. The diagram for such a process is shown in Figure 33.1*a*, where the circle denotes the inner part of the diagram, and Q' stands for the 4-momentum of particle M in the final state.

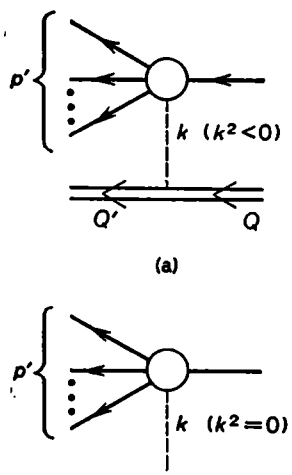

Figure 33.1

Under certain conditions (see below), the calculation of the above-noted process can be reduced to that of a simpler process (a photoprocess) in which a real photon ($k^2 = 0$) participates (see Figure 33.1*b*). Indeed, if particle M is an ultrarelativistic one (in a certain reference frame), then the electromagnetic field created by this particle closely resembles a plane-wave field, that is, is almost transverse (see (14.25) and (14.26)). Therefore, the virtual photon ($k^2 < 0$) closely resembles the real photon ($k^2 \simeq 0$), and the field created by particle M proves to be equivalent (hence the name given to the method) to a flux of real photons with a definite distribution over frequencies ω. Suppose that $dN(\omega)$ is the number of photons in the interval from ω to $\omega + d\omega$, and $\sigma^{ph}(\omega)$ is the cross section of the photoprocess in which a real photon of frequency ω participates (Figure 33.1*b*). The cross section of the initial process (Figure 33.1*a*) is then approximately given by the following formula:

$$d\sigma = \int \sigma^{ph}(\omega)\, dN(\omega), \tag{33.1}$$

where integration is over the spectrum of the equivalent photons dN.

We will now derive formula (33.1), which is known as the Weizsäcker-Williams formula, and obtain the spectrum dN*.

According to the general theory (Section 26), the matrix element of the process under consideration is

$$A_{fi} = Ze^2 \frac{4\pi}{k^2} (j^\mu J_\mu), \tag{33.2}$$

where J_μ is the current representing the transition of particle m into the ensemble of particles with momentum p', and j^μ is the current representing the transition of particle M, which we assume to

* We follow the paper by V. N. Gribov, V. A. Kolkunov, L. B. Okun, and V. M. Shekhter, *Zh. Eksper. Teor. Fiz.* 41, 1839 (1961).

be a point-like spinless (scalar) particle with charge Ze. For scalar particles the transition current

$$j^{\mu}_{fi} = i \int d^4x \, [\varphi^*_f \partial^{\mu} \varphi_i - (\partial^{\mu} \varphi^*_f) \, \varphi_i] \, e^{ikx},$$

with wave functions $\varphi_i = e^{-iQx}$ and $\varphi_f = e^{-iQ'x}$, is reduced to

$$\begin{aligned} j^{\mu}_{fi} &= (2\pi)^4 \, \delta_4 \, (Q' + k - Q) \, j^{\mu}, \\ j^{\mu} &= (Q + Q')^{\mu} = (2Q - k)^{\mu}. \end{aligned} \tag{33.3}$$

Note that the delta functions expressing conservation laws have been excluded from the reduced matrix element A_{fi}, so that $k = Q - Q'$ in (33.2), and we must substitute into (33.2) the current j^{μ} given by (33.3). Then we obtain

$$A_{fi} = \frac{4\pi Z e^2}{k^2} (2Q - k)^{\mu} J_{\mu}. \tag{33.4}$$

Knowing (33.4), we employ the general rules (Section 27) and find the cross section:

$$\begin{aligned} d\sigma = \frac{(4\pi Z e^2)^2}{4Ik^4} \frac{d^3Q'}{(2\pi)^3 \, Q'_0} \int (2\pi)^4 \, \delta_4 \, (p + Q - p' - Q') \\ \times (2Q - k)^{\mu} \, (2Q - k)^{\nu} \, J^{\dagger}_{\mu} J_{\nu} \prod_a \frac{d^3p'_a}{2p'_{a0} \, (2\pi)^3}, \end{aligned} \tag{33.5}$$

where the subscript a numbers the particle in the ensemble with $p' = \sum\limits_a p'_a$, and

$$I = [(pQ)^2 - p^2Q^2]^{1/2} \tag{33.6}$$

is the invariant flux.

Averaging over the polarizations of particle m and summing over the polarizations and momenta of the particles with p'_a, we can find, via (33.5), the cross section in the form

$$d\sigma = -\frac{4\pi Z^2 e^2}{4Ik^4} (2Q - k)^{\mu} \, (2Q - k)^{\nu} \, T_{\mu\nu} \frac{d^3Q'}{(2\pi)^3 \, 2Q'_0}, \tag{33.7}$$

where $T_{\mu\nu}$ is the symmetric tensor obtained from $e^2 J^{\dagger}_{\mu} J_{\nu}$ as a result of the above-noted operations of averaging and summing.

The general form of the $T_{\mu\nu}$ tensor satisfying the gauge invariance condition (current conservation)

$$T_{\mu\nu} k^{\nu} = T_{\mu\nu} k^{\mu} = 0, \tag{33.8}$$

is

$$T_{\mu\nu} = a \left[\frac{k^2}{pk} \, p_{\mu} p_{\nu} + (pk) \, g_{\mu\nu} - p_{\mu} k_{\nu} - p_{\nu} k_{\mu} \right] + b \, (k^2 g_{\mu\nu} - k_{\mu} k_{\nu}), \tag{33.9}$$

where a and b are functions of the invariants k^2, p^2, and pk. Here we have allowed for the fact that $T_{\mu\nu}$ can depend, by its definition, only on the 4-momenta k and p.

Calculating the contraction $T_{\mu\nu}(2Q-k)^\mu (2Q-k)^\nu = 4Q^\mu Q^\nu T_{\mu\nu}$ by employing (33.8) and substituting the result into (33.7), we arrive at the following formula for the cross section:

$$d\sigma = -\frac{4\pi Z^2 e^2}{I}\frac{(pQ)^2}{(pk)\,k^2}\left\{a\left[1+\frac{(pk)^2\,Q^2}{(pQ)^2\,k^2}-\frac{(pk)}{(pQ)}\right]\right.$$
$$\left.+\,b\left(Q^2-\frac{k^2}{4}\right)\frac{(kp)}{(pQ)^2}\right\}\frac{d^3Q'}{(2\pi)^3\,2Q'_0}. \qquad (33.10)$$

Now let us turn to the diagram in Figure 33.1*b* representing the photoprocess. The matrix element of this process is

$$A_{fi}^{\mathrm{ph}} = -ie\sqrt{4\pi}\,e^\mu \overset{0}{J}_\mu, \quad \overset{0}{J}_\mu = J_\mu\,|_{k^2=0}, \qquad (33.11)$$

where e^μ is the polarization 4-vector of the real photon.

Let us write the cross section of the photoprocess with account taken of (33.11),

$$\sigma^{\mathrm{ph}} = \frac{e^{*\mu}e^\nu}{4(pk)}\int (2\pi)^4\,\delta_4(p+k-p')\,4\pi e^2 \overset{0}{J}{}^+_\mu \overset{0}{J}_\nu \prod_a \frac{d^3p'_a}{(2\pi)^3\,2p'_{a0}}$$
$$= -\frac{e^{*\mu}e^\nu}{4(pk)}\overset{0}{T}_{\mu\nu}, \quad \overset{0}{T}_{\mu\nu} = T_{\mu\nu}\,|_{k^2=0}. \qquad (33.12)$$

Note that here $\overset{0}{T}_{\mu\nu}$ is the same tensor (33.9) but taken for $k^2 = 0$. Averaging over the photon polarizations according to

$$\overline{e^{*\mu}e^\nu} = -\frac{1}{2}g^{\mu\nu}$$

and allowing for (33.9), we get

$$\overset{0}{T}_{\mu\nu}g^{\mu\nu} = 2\overset{0}{a}(pk), \quad \overset{0}{a} = a\,|_{k^2=0},$$

and as a result the photoprocess cross section (33.12) is reduced to

$$\sigma^{\mathrm{ph}} = \frac{1}{4}\overset{0}{a}. \qquad (33.13)$$

Suppose the following conditions are met:

$$|k^2| \ll p^2 = m^2, \quad |k^2| \ll (kp),$$
$$(pQ)^2 \gg p^2Q^2 = m^2M^2, \qquad (33.14)$$
$$(pQ) \gg (pk),$$

where for the sake of definiteness we have assumed that $M > m$. Then in the cross section (33.10) we can put

$$a \simeq \overset{0}{a},\ I \simeq pQ$$

and ignore the term proportional to b. Allowing for (33.13), we can reduce the formula for the cross section to

$$d\sigma = \sigma^{\mathrm{ph}} \frac{Z^2e^2}{\pi^2} \frac{(pQ)}{(pk)\,k^2} \left[1 - \frac{Q^2\,(pk)^2}{(-k^2)\,(pQ)^2}\right] \frac{d^3Q'}{Q_0'}, \qquad (33.15)$$

where σ^{ph} is the photoprocess cross section.

Let us go over to the invariant variables $(-k^2)$ and (pk) in the phase volume d^3Q'/Q_0'. It is expedient at this point to go over to the center-of-mass system involving particles m and M (in this system $\mathbf{p} + \mathbf{Q} = 0$). We denote the angle between $\mathbf{Q}$ and $\mathbf{Q}'$ by θ and the azimuthal angle of $\mathbf{Q}'$ in a plane perpendicular to $\mathbf{Q}$ by φ. We have

$$\frac{d^3Q'}{Q_0'} = |\,\mathbf{Q}'\,|\,dQ_0'\,d\varphi\,d\cos\theta, \qquad (33.16)$$

where we have allowed for the fact that $|\,\mathbf{Q}'\,|\,d\,|\,\mathbf{Q}'\,| = Q_0'dQ_0'$. Moreover,

$$k^2 = (Q - Q')^2 = 2M^2 - 2Q_0Q_0' + 2\,|\,\mathbf{Q}\,|\,|\,\mathbf{Q}'\,|\cos\theta,$$
$$s \equiv (p + k)^2 = (p + Q - Q')^2$$
$$= M^2 + (p_0 + Q_0)^2 - 2\,(p_0 + Q_0)\,Q_0'.$$

The Jacobian of the $(\cos\theta, Q_0') \to (s, k^2)$ transformation is

$$\frac{\partial\,(s,\ k^2)}{\partial\,(\cos\theta,\ Q_0')} = -4\,(p_0 + Q_0)\,|\,\mathbf{Q}\,|\,|\,\mathbf{Q}'\,|\,Q_0',$$

whereby (33.16) is reduced to

$$\frac{d^3Q'}{Q_0'} = \frac{1}{4} \frac{ds\,d\,(-k^2)\,d\varphi}{|\,\mathbf{Q}\,|\,(p_0 + Q_0)} \to \frac{\pi}{(pQ)}\,d\,(-k^2)\,d\,(pk). \qquad (33.17)$$

Here we have carried out integration with respect to φ, which yielded 2π, and employed the relationships $ds = d\,(p^2 + k^2 + 2pk) = 2d\,(pk)$ and $|\,\mathbf{Q}\,|\,(p_0 + Q_0) = [(pQ)^2 - p^2Q^2]^{1/2} \simeq pQ$ (since $\mathbf{p} + \mathbf{Q} = 0$ and (33.14) is met).

Substituting (33.17) into (33.15) yields

$$d\sigma = \sigma^{\mathrm{ph}} \frac{Z^2e^2}{\pi} \left[1 - \frac{Q^2\,(pk)^2}{(-k^2)\,(pQ)^2}\right] \frac{d\,(pk)}{(pk)} \frac{d\,(-k^2)}{(-k^2)}. \qquad (33.18)$$

We integrate with respect to k^2 within the following limits:

$$(-k^2)_{\min} = \frac{Q^2\,(pk)^2}{(pQ)^2} \leqslant (-k^2) \leqslant (-k^2)_{\max} = m^2, \qquad (33.19)$$

where the upper limit corresponds to the boundary within which the first inequality in (33.14) is satisfied, and the lower limit corresponds to the condition that the cross section (33.18) be nonnegative (precisely, that the expression within the square brackets be nonnegative). As a result we obtain

$$\int_{(-k^2)_{\min}}^{(-k^2)_{\max}} \frac{d(-k^2)}{(-k^2)}[\ldots] = \int_{1j}^{x_{\max}} \frac{dx}{x}\left(1-\frac{1}{x}\right) = \left(\ln x + \frac{1}{x}\right)\Big|_1^{x_{\max}}$$

$$= \ln x_{\max} + \frac{1}{x_{\max}} - 1 \simeq \ln x_{\max}$$

provided that

$$x_{\max} \equiv \frac{p^2 (pQ)^2}{Q^2 (pk)^2} \gg 1, \quad \ln x_{\max} \gg 1, \tag{33.20}$$

that is, in the so-called logarithmic approximation. Inequalities (33.20) agree with (33.14), obviously. We note also that the upper limit in (33.19), $(-k^2)_{\max}$, is actually defined only in its order of magnitude, but in the logarithmic approximation (33.20) this is unimportant.

Let us introduce parameter μ and a new parameter $\varkappa$ instead of (pk) thus:

$$\mu = (pQ)/mM, \quad \varkappa = 2\,(pk)/m^2. \tag{33.21}$$

Allowing for (33.18), (33.20), and (33.21), we arrive at the following expression for the cross section in the equivalent photon approximation:

$$d\sigma = \int \sigma^{\mathrm{ph}}(\varkappa)\, dN(\varkappa), \tag{33.22}$$

where the spectrum of equivalent photons is given by the formula

$$dN = \frac{2}{\pi} Z^2 e^2 \ln\frac{\mu}{\varkappa}\,\frac{d\varkappa}{\varkappa}. \tag{33.23}$$

The conditions that approximation (33.22) must meet so as to be valid will be written in terms of μ and $\varkappa$ thus (see (33.14) and (33.20)):

$$|k^2| \ll m^2, \quad |k^2| \ll m^2\varkappa, \quad \mu \gg 1, \quad \mu/\varkappa \gg m/M, \quad \ln(\mu/\varkappa) \gg 1. \tag{33.24}$$

Let us discuss the physical meaning of these conditions. Most important, they specify a kinematic region within which the differential cross section is expressed in terms of the photoprocess cross section via (33.15). If this region provides the main contribution to the cross section (for this to be true the functions a and b in (33.9) must not increase with $|k^2|$) integration over the equivalent photon spectrum in (33.22) provides the leading logarithmic term in the total cross

section (the "large logarithm"), with the argument of the logarithm defined to within a factor of the order of unity.

The inequality $\mu \gg 1$ means that the relative velocity of the initial particles m and M is close to that of light (this is the reason why a virtual photon in this case is quasi-real, so to say). Indeed, in the system in which particle M is at rest ($\mathbf{Q} = 0$) we have $\mu = p_0/m \gg 1$, that is, the particle with mass m is ultrarelativistic. In the system in which particle m is at rest ($\mathbf{p} = 0$) particle M proves to be ultrarelativistic ($\mu = Q_0/M \gg 1$), $\varkappa = 2\omega/m$, and the condition $\mu/\varkappa \gg m/M$ means that $Q_0 \gg \omega$. Hence, we can ignore the recoil of particle M and assume that this particle is in uniform classical motion, in other words, is a classical source. This means that the equivalent photon spectrum (33.23) can be obtained also by employing methods of classical electrodynamics, namely, by expanding the almost transverse field of the uniformly moving ultrarelativistic particle in plane monochromatic waves [12].

The method of equivalent photons has found wide applications in estimating the cross sections of electrodynamical processes that involve high-energy particles. Lately, in a modified form, it has found applications in the theory of strong interactions, quantum chromodynamics, in calculations of the interactions between quarks and gluons.

34. Bremsstrahlung Emitted by Relativistic Electrons in Collisions with Nuclei

Let us study the emission of photons by electrons that collide with a massive nucleus. This process is described by the block diagram (a) in Figure 34.1, which is the sum of two diagrams, (b) and (c), repre-

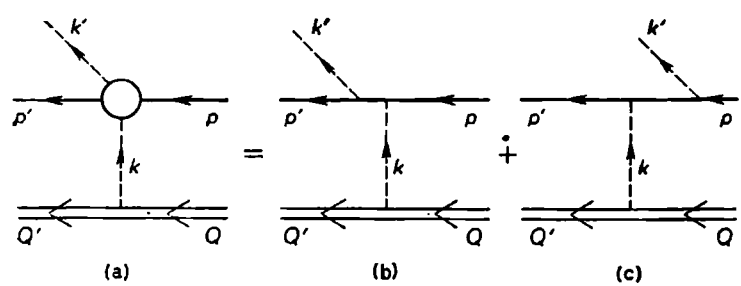

Figure 34.1

senting the emission of a bremsstrahlung photon by the final or initial electron, respectively. The reader can easily see that because the mass of the nucleus is large compared to the electron mass m ($M \gg m$) we can ignore any emission of a photon by the nucleus. Indeed,

even from dimensionality considerations it follows that the cross section σ_m of photon emission by a particle of mass m is proportional to the square of the classical radius of this particle, $r_0 = e^2/m$, whereby the ratio of the cross sections of photon emission by particles M and m is of the order of $(m/M)^2 \ll 1$.

34.1 Bremsstrahlung Generated by Ultrarelativistic Electrons

An exact calculation of the bremsstrahlung cross section for an arbitrary electron energy is a rather complex task. We will therefore restrict our discussion to the important case of an ultrarelativistic electron (in the reference frame in which the nucleus is at rest). In this case one is able to employ the approximate method of equivalent photons discussed in Section 33 (it can easily be verified that the applicability conditions (33.24) are indeed met).

The photoprocess corresponding to bremsstrahlung is the Compton scattering of photon k by an electron with initial momentum p (see Figure 34.1). The cross section of the photoprocess is given by (30.34):

$$\sigma^{\mathrm{ph}}(\varkappa) = \frac{2\pi r_0^2}{\varkappa} \int_0^{\varkappa} \frac{du}{(1+u)^2} \left[2 + \frac{u^2}{1+u} - 4\frac{u}{\varkappa}\left(1 - \frac{u}{\varkappa}\right)\right], \qquad (34.1)$$

$$u = (kk')/(kp'), \quad \varkappa = 2(kp)/m^2,$$

with $r_0 = e^2/m$.

According to (33.22) and (34.1), the bremsstrahlung cross section in the equivalent photon approximation assumes the form

$$\frac{d\sigma_{\mathrm{B}}}{du} = \int_u^{\infty} d\varkappa \frac{dN}{d\varkappa} \frac{d\sigma^{\mathrm{ph}}(\varkappa, u)}{du}, \qquad (34.2)$$

where $d\sigma^{\mathrm{ph}}/du$ corresponds to the integrand in (34.1), and the spectrum $dN(\varkappa)$ is given by (33.23).

Integration with respect to $\varkappa$ in (34.2) can be carried out in the logarithmic approximation in the following manner:

$$\int_u^{\infty} d\varkappa \ln\frac{\mu}{\varkappa} f(\varkappa) = -\ln\frac{\mu}{\varkappa} \int_{\varkappa}^{\infty} f(\varkappa)\, d\varkappa \Big|_{\varkappa=u}^{\infty} - \int_u^{\infty} \frac{f(\varkappa)}{\varkappa} d\varkappa$$

$$\simeq \ln\frac{\mu}{u} \int_u^{\infty} f(\varkappa)\, d\varkappa. \qquad (34.3)$$

Employing (34.3) and combining (34.1) with (34.2) we arrive at an invariant expression for the bremsstrahlung cross section:

$$d\sigma_{\mathrm{B}} = 4\alpha Z^2 r_0^2 \frac{du}{(1+u)^2}\left(\frac{u}{1+u} + \frac{4}{3u}\right)\ln\frac{\mu}{u}, \qquad (34.4)$$

with $\ln(\mu/u) \gg 1$ within the limits of applicability of the method, and Ze being the charge of the nucleus.

Let us write (34.4) in the laboratory system, in which the nucleus is at rest ($\mathbf{Q} = 0$) and the electron is ultrarelativistic $\varepsilon = p_0 \gg m$. In such a system we have (see (33.21))

$$u = \frac{\omega'}{\varepsilon - \omega'}, \quad \varepsilon' = \varepsilon - \omega', \quad \frac{du}{(1+u)^2} = d\omega',$$

where ω' is the frequency of the bremsstrahlung photon, and ε' is the energy of the final electron. The bremsstrahlung cross section assumes the form

$$d\sigma_B = 4Z^2\alpha r_0^2 \frac{d\omega'}{\omega'} \frac{\varepsilon'}{\varepsilon} \left(\frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon} - \frac{2}{3} \right) \ln \frac{\varepsilon\varepsilon'}{m\omega'}, \qquad (34.5)$$

where $\ln(\varepsilon\varepsilon'/m\omega') \gg 1$. In particular, the bremsstrahlung cross section for soft photons ($\omega' \to 0$, $\varepsilon' \simeq \varepsilon$) assumes the form

$$d\sigma_B = \frac{16}{3} Z^2\alpha r_0^2 \frac{d\omega'}{\omega'} \ln \frac{\varepsilon^2}{m\omega'}. \qquad (34.5a)$$

Note that the cross section given by (34.5a) is divergent when $\omega' \to 0$, which leads to the total cross section being divergent at small frequencies as well. The reason for this is that for low frequencies the perturbation-theory approach proves to be invalid since the probability of emission of two or more soft photons becomes higher than that of emission of a single photon. A detailed analysis of the emission of soft photons was carried out earlier (see Section 25).

A quantity that is often introduced in this connection is the radiative energy loss cross section, which is related to the bremsstrahlung cross section $d\sigma_B$ through the following formula:

$$\sigma_{emis} = \frac{1}{\varepsilon} \int_0^{\varepsilon - m} \omega' \frac{d\sigma_B}{d\omega'} d\omega', \qquad (34.6)$$

and the quantity $\varepsilon\sigma_{emis}$ is known as the effective bremsstrahlung factor. Substituting cross section (34.5) into (34.6) and integrating with respect to ω' in the logarithmic approximation (see (34.3)), we arrive at the following formula for the radiative energy loss cross section involving an ultrarelativistic electron:

$$\sigma_{em's} = 4Z^2\alpha r_0^2 \ln(\varepsilon/m). \qquad (34.7)$$

Suppose a high-energy electron is moving through a medium containing N nuclei (or ions) per unit volume. Then the electron energy lost

because of bremsstrahlung per unit length of the electron's path in the medium is

$$-\frac{d\varepsilon}{dx} = N\varepsilon\sigma_{\text{emis}} \tag{34.8}$$

and the electron loses its entire energy over distances of the order of

$$l_{\text{rad}} \sim 1/N\sigma_{\text{emis}}, \tag{34.9}$$

where l_{rad} is called the radiation length.

34.2 Electron-Positron Pair Creation by a Photon at a Nucleus

The diagrams representing this process are shown in Figure 34.2. We will not take up the general case of incident photons with arbitrary energies. Rather we will consider the calculation of the total cross

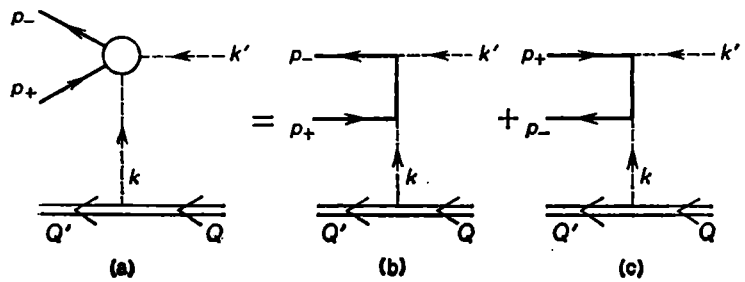

Figure 34.2

section of the process in which a high-energy photon with energy $\omega' \gg m$ (in the laboratory system, where the nucleus is at rest) interacts with a nucleus as a result of which an electron-positron pair is created. The calculation will be carried out in the equivalent photon approximation. The corresponding photoprocess here is the creation of an electron-positron pair e^-e^+ by two photons with momenta k' and k (we discussed this process in Section 32).

In the specified approximation, the total cross section can be written in the form (cf. (34.2))

$$\sigma_{\text{creat}} = \int_1^\infty du \int_u^\infty \frac{dN(\varkappa_1)}{d\varkappa_1} \frac{d\sigma^{\text{ph}}_{\text{creat}}(\varkappa_1, u)}{du} d\varkappa_1, \tag{34.10}$$

where $d\sigma^{\text{ph}}_{\text{creat}}$ is the cross section of two-photon pair creation (32.11), and the variable u and the paramter $\varkappa_1$ are such (see (32.5)) that

$$u = \frac{(kk')^2}{4(p_+k)(p_-k')}, \quad \varkappa_1 = \frac{kk'}{2m^2}.$$

The equivalent photon spectrum $dN(\varkappa_1)$ is given by formula (33.23) if we substitute $\varkappa_1$ for $\varkappa$ with parameter μ given by the formula $\mu = k'Q/mM$.

Integration in (34.10) with respect to $\varkappa_1$ and u in the logarithmic approximation (see (34.3)) yields

$$\sigma_{\text{creat}} = \frac{28}{9} Z^2 \alpha r_0^2 \ln \mu, \tag{34.11}$$

and in the laboratory system ($\mathbf{Q} = 0$) we have $\mu = \omega'/m$. The formula we have just obtained is valid for $\ln \mu \gg 1$.

At high energies the cross section of the process increases with the photon energy as a logarithmic function. For this reason pair creation is the main mechanism by which high-energy photons lose their energy as they travel through matter (the photoeffect and the Compton effect (see Section 30) compete with this process, but their cross sections rapidly decrease in the high-energy range for photons).

35. Vacuum Effects. Renormalization in Quantum Electrodynamics

Up till now we have considered electrodynamical effects only within the framework of low-order perturbation theory. Now we wish to discuss the effect of correction terms of a higher order (the so-called radiative corrections). We will not expound the general theory (the interested reader can refer to [12, 9]). Instead we will illustrate the conclusions of the theory using one simple example, the scattering of electrons in an external field.

35.1 Scattering of an Electron in an External Field

This effect manifests itself in first-order perturbation theory. The respective diagram is (a) in Figure 35.1, where the "x" denotes the line of the external field (a virtual photon). The first-order S-matrix element is

$$S_{fi}^{(1)} = -ie \int \bar{\psi}_{p'}(x)\, \gamma^\mu A_\mu(x)\, \psi_p(x)\, d^4x, \tag{35.1}$$

where ψ_p and $\bar{\psi}_{p'}$ are the wave functions of the initial and final electrons with 4-momenta p and p' and normalized in the usual manner (see Section 21):

$$\psi_p(x) = \frac{1}{\sqrt{2\varepsilon V}} u(p)\, e^{-ipx}, \quad \bar{\psi}_{p'}(x) = \frac{1}{\sqrt{2\varepsilon' V}} \bar{u}(p')\, e^{ip'x}, \tag{35.2}$$

and $A_\mu(x)$ is the 4-potential of the external (classical) field, which is conveniently written in the form of a Fourier integral:

$$A_\mu(x) = \int \frac{d^4q}{(2\pi)^4} e^{-iqx} A_\mu(q), \tag{35.3}$$

so that the Fourier transform of the potential is

$$A_\mu(q) = \int d^4x e^{iqx} A_\mu(x). \tag{35.4}$$

Substituting (35.2) and (35.3) into (35.1), we arrive at the following formula after integration with respect to x:

$$S_{fi}^{(1)} = -ie \int \frac{d^4q}{(2\pi)^4} (2\pi)^4 \delta_4(p+q-p') \bar{u}(p') \times \gamma^\mu A_\mu(q) u(p) (2\varepsilon V 2\varepsilon' V)^{-1/2}. \tag{35.5}$$

The integral with respect to q can easily be evaluated because of the presence of a delta function, and we arrive at the reduced amplitude (see Section 27)

$$A_{fi}^{(1)} = -e\bar{u}(p') \gamma^\mu A_\mu(q) u(p), \tag{35.6}$$

which is related to the S-matrix element through the formula

$$S_{fi}^{(1)} = \frac{iA_{fi}^{(1)}}{(2\varepsilon V 2\varepsilon' V)^{1/2}}, \tag{35.7}$$

with $q = p' - p$ in (35.6) being the momentum transfer.

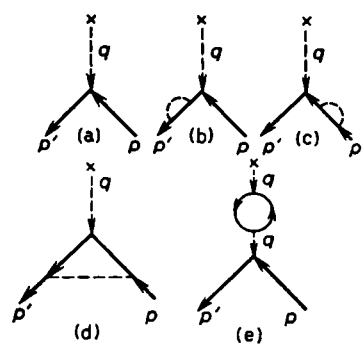

Figure 35.1

Note that (35.7) does not contain a delta function, which is a reflection of the fact that the 4-momentum of the electron is not conserved in an external field (cf. (26.7) and (27.5)). Therefore the transition from the amplitude of the process to the probability differs somewhat from that discussed in Section 27. We will restrict our discussion to the case of a timeindependent external field, when $A_\mu(x) = A_\mu(\mathbf{r})$ and the four-dimensional Fourier transform of the potential,

$$A_\mu(q) = 2\pi\delta(q_0) a_\mu(\mathbf{q}), \tag{35.8}$$

is expressed in terms of the appropriate transform of the spatial distribution thus:

$$a_\mu(\mathbf{q}) = \int d^3x e^{-i\mathbf{q}\cdot\mathbf{r}} A_\mu(\mathbf{r}). \tag{35.8a}$$

Substitution of (35.8) into (35.6) brings (35.7) to the following form:

$$S_{fi}^{(1)} = i2\pi\delta(q_0)\frac{a_{fi}}{(2\varepsilon V 2\varepsilon' V)^{1/2}}, \tag{35.9}$$

where we have introduced the new amplitude

$$a_{fi} = -e\bar{u}(p')\gamma^{\mu}a_{\mu}(\mathbf{q})u(p), \tag{35.10}$$

and the delta function $\delta(q_0) = \delta(\varepsilon' - \varepsilon)$ expresses the conservation of the electron's energy during scattering in a time independent external field.

Now from the matrix element (35.9) we go over to the transition rate via (27.3). Instead of the square of the four-dimensional delta function, (27.2), there appears the square of the one-dimensional delta function $\delta(q_0)$, which square must be interpreted thus:

$$|\delta(q_0)|^2 = \frac{T}{2\pi}\delta(q_0),$$

where, like in Section 27, T is a large time interval. As a result we get the transition rate:

$$dw = 2\pi\delta(\varepsilon - \varepsilon')\,|\,a_{fi}\,|^2\frac{1}{2\varepsilon V}\frac{d^3p'}{(2\pi)^3 2\varepsilon'}, \tag{35.11}$$

which, in contrast to (27.8), depends on the normalization volume V. The quantity that has physical meaning in this case is the scattering cross section $d\sigma$, which according to (27.9) is obtained by dividing (35.11) by the flux of the incident electrons (cf. (27.10)) $j = v/V$, where $v = |\,\mathbf{p}\,|/\varepsilon$ is the speed of an initial electron. The result is

$$d\sigma = 2\pi\delta(\varepsilon' - \varepsilon)\,|\,a_{fi}\,|^2\frac{1}{2\,|\,\mathbf{p}\,|}\frac{d^3p'}{(2\pi)^3 2\varepsilon'}. \tag{35.12}$$

Let us consider the important case of scattering in a Coulomb field of a charge Ze. This field is described by the 4-potential

$$A^{\mu}(\mathbf{r}) = (Ze/r, 0, 0, 0). \tag{35.13}$$

Substituting (35.13) into (35.8a), we get

$$a^0(\mathbf{q}) = \frac{4\pi Ze}{\mathbf{q}^2}, \quad \mathbf{a}(\mathbf{q}) = 0, \tag{35.14}$$

and the amplitude (35.10) assumes the form

$$a_{fi} = -\frac{4\pi Ze^2}{\mathbf{q}^2}\bar{u}(p')\gamma^0 u(p). \tag{35.15}$$

Let us calculate the scattering cross section for nonpolarized electrons. Averaging the square of amplitude (35.15) over the initial

spin states of the electron and summing it over the final spin states according to the general rules (see (27.19)), we get

$$\frac{1}{2}\sum_{\text{spin}} |a_{fi}|^2 = \frac{1}{2}\left(\frac{4\pi Ze}{\mathbf{q}^2}\right)^2 \operatorname{Tr}\{(\gamma p + m)\gamma^0(\gamma p' + m)\gamma^0\}. \quad (35.16)$$

The calculation of the trace can be simplified if we note that $\gamma^0(\gamma p')\gamma^0 = \gamma\tilde{p}$, where $\tilde{p} = (\varepsilon', -\mathbf{p}')$, with $\varepsilon' = \varepsilon$. The calculation amounts to finding

$$\frac{1}{4}\operatorname{Tr}[(\gamma p + m)(\gamma\tilde{p} + m)] = m^2 + (\tilde{p}p) = 2\varepsilon^2 - \frac{\mathbf{q}^2}{2}, \quad (35.17)$$

where we have used formula (30.9) and the fact that $\tilde{p}p = \varepsilon^2 + \mathbf{p}\cdot\mathbf{p}' = \varepsilon^2 + \mathbf{p}\cdot(\mathbf{p} + \mathbf{q}) = 2\varepsilon^2 + \mathbf{p}\cdot\mathbf{q}$, with $\mathbf{p}\cdot\mathbf{q} = -\mathbf{q}^2/2$, since $q = p' - p$ and, hence, $p'^2 = m^2 = (q + p)^2 = q^2 + 2qp + m^2$, $q^2 = -\mathbf{q}^2$, and $qp = -\mathbf{q}\cdot\mathbf{p}$, in view of the fact that $q^0 = 0$.

Combining (35.16), (35.17), and (35.12), we arrive at the following formula for the scattering cross section

$$\frac{d\sigma}{d\Omega} = 4(Ze^2)^2\frac{\varepsilon^2}{\mathbf{q}^4}\left(1 - \frac{q^2}{4\varepsilon^2}\right), \quad (35.18)$$

where $d\Omega$ is the solid angle element in the direction of the momentum $\mathbf{p}'$ of the scattered electron. Let us introduce the scattering angle: $\theta = (\widehat{\mathbf{p}, \mathbf{p}'})$. Then the square of the momentum transfer (from the external field to the electron) is

$$\mathbf{q}^2 = (\mathbf{p}' - \mathbf{p})^2 = 4\mathbf{p}^2\sin^2\frac{\theta}{2}, \quad (35.19)$$

where we have allowed for the fact that $|\mathbf{p}'| = |\mathbf{p}|$. Combining (35.19) with (35.18), we arrive at Mott's formula for the cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z\alpha}{2|\mathbf{p}|v}\right)^2 \sin^{-4}\frac{\theta}{2}\left(1 - v^2\sin^2\frac{\theta}{2}\right). \quad (35.20)$$

In the nonrelativistic limit, this formula transforms into the classical Rutherford formula.

35.2 Radiative Corrections to the Scattering Amplitude

Formula (35.20) was obtained within the framework of first-order perturbation theory by employing amplitude (35.10), which can now be compactly written as

$$A^{(1)} = -e\bar{u}(p')\gamma^\mu a_\mu u(p). \quad (35.21a)$$

Let us now allow for corrections to this amplitude that are of the order of e^2. The corresponding diagrams in Figure 35.1 are (b), (c), (d), and (e). According to Feynman's rules, we can write expressions for the corresponding amplitudes, denoting them by $A^{(3)}_{(b)}$, $A^{(3)}_{(c)}$, $A^{(3)}_{(d)}$, and $A^{(3)}_{(e)}$:

$$A^{(3)}_{(b)} = -e\bar{u}(p')\,M(p')\,S(p')\,\gamma^\mu a_\mu u(p), \tag{35.21b}$$

$$A^{(3)}_{(c)} = -e\bar{u}(p')\,\gamma^\mu a_\mu S(p)\,M(p)\,u(p), \tag{35.21c}$$

$$A^{(3)}_{(d)} = -e\bar{u}(p')\,\Lambda^\mu(p', p)\,a_\mu u(p), \tag{35.21d}$$

$$A^{(3)}_{(e)} = -e\bar{u}(p')\,\gamma^\mu \delta a_\mu u(p). \tag{35.21e}$$

Here we have introduced the following quantities: the mass operator

$$M(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\mu S(p-k)\,\gamma^\nu i D_{\mu\nu}(k)$$

$$= -4\pi i e^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\mu\,\frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma_\mu\,\frac{1}{k^2+i\varepsilon}; \tag{35.22}$$

the vertex operator

$$\Lambda^\mu(p', p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\alpha S(p'-k)\,\gamma^\mu S(p-k)\,\gamma^\beta i D_{\alpha\beta}(k)$$

$$= -4\pi i e^2 \int \frac{d^4k}{(2\pi)^4}\,\gamma^\nu\,\frac{\hat{p}'-\hat{k}+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma^\mu\,\frac{\hat{p}-\hat{k}+m}{(p-k)^2-m^2+i\varepsilon}\,\gamma_\nu\,\frac{1}{k^2+i\varepsilon}; \tag{35.23}$$

and the correction δa^μ (cf. (35.21e) and (35.21a)) to the external-field potential a^μ (35.14)

$$\delta a_\mu(q) = \frac{1}{4\pi} D_{\mu\nu}(q)\,P^{\nu\lambda}(q)\,a_\lambda(q) = \frac{1}{q^2+i\varepsilon}\,P_{\mu\nu}(q)\,a^\nu(q), \tag{35.24}$$

where

$$P^{\mu\nu}(q) = (-1)(-4\pi i)(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\,[\gamma^\mu i S(k)\,\gamma^\nu i S(k-q)]$$

$$= +4\pi i e^2 \int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\mu\,\frac{\hat{k}+m}{k^2-m^2+i\varepsilon}\,\gamma^\nu\,\frac{\hat{k}-\hat{q}+m}{(k-q)^2-m^2+i\varepsilon}\right] \tag{35.25}$$

is known as the polarization operator, and in writing this operator we have, via Feynman's rules (see Section 26), allowed for the common factor -1 introduced by the electron loop in diagram (e) in Figure 35.1. We have also used the following notations in (35.21)-(35.25): $\hat{a} \equiv \gamma^\mu a_\mu$, $S(p) \equiv -iS_c(p) = (\hat{p}+m)/(p^2-m^2+i\varepsilon)$ is the electron propagator, and $D_{\mu\nu}(k) \equiv -iD^c_{\mu\nu}(k) = 4\pi g_{\mu\nu}/(k^2+i\varepsilon)$,

is the photon propagator (see (26.8) and (26.9)). Here and in what follows all the bispinor indices are dropped, for the sake of brevity.

Thus, the amplitudes $A_i^{(3)}$, with i = (b), (c), (d), and (e), are expressed in terms of the mass operator $M(p)$, the vertex operator $\Lambda^\mu(p', p)$, and the polarization operator $P^{\mu\nu}(q)$ defined by the integrals in (35.22), (35.23), and (35.25), respectively. However, there is a principal difficulty in all this: all these integrals are divergent when k tends to ∞, which follows immediately from counting the exponents k in the numerators and denominators of the integrands, namely, M diverges linearly, Λ^μ logarithmically, and $P^{\mu\nu}$ quadratically.

Thus, a formal perturbation calculation of the corrections to the first-order amplitude (35.21a) leads to a physically meaningless result. Note that formula (35.20) for the scattering cross section, which is obtained from formula (35.21a) for the amplitude, is in good agreement with the experimental data. This only means that a correct incorporation of the higher-order perturbation-theory terms must yield small (instead of infinite!) corrections to the cross section (35.20).

The crux of the problem lies in the fact that in building the perturbation theory in Section 26 we actually assumed that the electron is "bare", that is, not interacting with the electromagnetic vacuum; the perturbation was "switched on" at some moment and then "switched off". For this reason the constants e and m in the equations of quantum electrodynamics (see Section 23) have the meaning of the charge and mass of a "bare" (nonphysical) electron, while the real (or physical) electron has a charge e_0 and mass m_0 that differ from the "bare" values e and m (see also Section 24). It is the parameters e_0 and m_0 that are measured in experiments, and this implies that the results of all calculations to be compared with the experimental data must be expressed in terms of these parameters. The process of excluding the nonphysical quantities e and m via relations of the type $e = e(e_0, m_0)$ and $m = m(e_0, m_0)$ and of going over to the observables e_0 and m_0 is known as renormalization. It has been found that if the electron charge and mass are renormalized, the calculation of all observables in any order of the perturbation theory yields finite quantities. A theory that possesses this property is known as renormalizable, and quantum electrodynamics proves to be a theory of exactly this type, which may be rigorously proved [9].

35.3 Renormalization of Mass

The sum of amplitudes (35.21a), (35.21b), and (35.21c), to which amplitudes the diagrams (a), (b), and (c) in Figure 35.1 correspond, is the amplitude

$$A_m = A^{(1)} + A^{(3)}_{(b)} + A^{(3)}_{(c)} = -e\bar{U}(p')\gamma^\mu a_\mu U(p). \qquad (35.26)$$

This amplitude differs from $A^{(1)}$ in that the bispinors $u(p)$ and $\bar{u}(p')$ in it are replaced by the effective bispinors

$$U(p) = u(p) + S(p) M(p) u(p),$$
$$\bar{U}(p') = u(p') + \bar{u}(p') M(p') S(p'). \tag{35.27}$$

To these new quantities we can assign the so-called effective electron lines (see Figure 35.2, which illustrates the first equation in (35.27)). A loop (or "bubble") in such diagrams corresponds to the operator

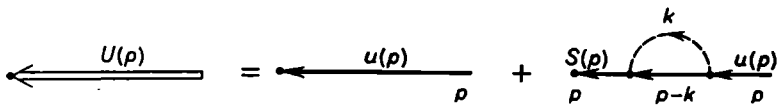

Figure 35.2

$M(p)$. Now let us discuss the physical meaning of this operator. Not striving for mathematical rigor, we will deal with infinite quantities as though they were finite. Applying $\hat{p} - m$ to $U(p)$ and allowing for the fact that $(\hat{p} - m) u(p) = 0$ and $(\hat{p} - m) S(p) = 1$, we obtain

$$(\hat{p} - m) U(p) = Mu(p).$$

Moreover, we note that allowing only for corrections of the order of e^2 to the main amplitude $A^{(1)}$, we can put $Mu \simeq MU$ and arrive at the following equation for $U(p)$:

$$(\hat{p} - m - M(p)) U(p) = 0.$$

If we compare this equation with the one for the common bispinor (of the "bare" electron) $(\hat{p} - m) u(p) = 0$, we will see that $M(p)$ describes the shift in the mass of the "bare" electron caused by the interaction of the electron with the electromagnetic vacuum.

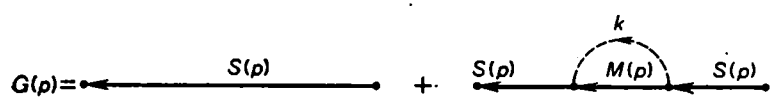

Figure 35.3

We can easily convince ourselves that if we allow for a correction proportional to e^2 to the electron propagator (see Figure 35.3), we are forced to perform the following substitution:

$$\frac{i}{\hat{p}-m} \to \frac{i}{\hat{p}-m} + \frac{i}{\hat{p}-m}(-iM(p))\frac{i}{\hat{p}-m} = \frac{i}{\hat{p}-m-M(p)} + O(\alpha^2), \tag{35.28}$$

which illustrates the meaning of $M(p)$ quite clearly. The modified propagator $G(p) = [\hat{p} - m - M(p)]^{-1}$ obviously satisfies the equation

$$[\hat{p} - m - M(p)]\, G(p) = 1,$$

which is similar to the equation $(\hat{p} - m)\, S = 1$ and can be written (within the accuracy we have allowed for here), by virtue of (35.28), in the form

$$G(p) = S(p) + S(p)\, M(p)\, S(p) = \frac{1}{\hat{p} - m - M(p)}. \qquad (35.29)$$

Now let us consider the mass operator $M(p)$ in greater detail and discuss how precisely it describes the correction to the electron mass. The integral (35.22) is divergent. To regularize it, we introduce the following substitution for the photon propagator:

$$\frac{1}{k^2} \to \frac{1}{k^2 - \lambda^2} \frac{-\Lambda^2}{k^2 - \Lambda^2}, \qquad (35.30)$$

with $\lambda \ll m$ and $\Lambda \gg m$. The quantity λ has the meaning of a small photon mass and is introduced to eliminate the "infrared" divergence for small values of k^2 in the integral regularized via (35.30) (the initial integral (35.22) does not contain it). This divergence is fictitious, since if observables are calculated correctly, their dependence on λ vanishes (see below). Parameter Λ is needed for the integral to converge as $k \to \infty$. Note that the substitution (35.30) is equivalent to a restriction of the domain of integration with respect to k by the condition $\lambda^2 \leqslant k^2 \leqslant \Lambda^2$ in (35.22).

Let us denote the regularized mass operator by $\overline{M}(p)$. It is defined thus:

$$\overline{M}(p) = -8\pi i e^2 \int \frac{2m - \hat{p} + \hat{k}}{(p-k)^2 - m^2} \frac{1}{k^2 - \lambda^2} \left(\frac{-\Lambda^2}{k^2 - \Lambda^2}\right) \frac{d^4k}{(2\pi)^4}, \qquad (35.31)$$

where in going over from (35.22) to (35.31) we have employed (35.30) and the relationships $\gamma^\mu \gamma_\mu = 4$ and $\gamma^\mu \hat{p} \gamma_\mu = -2\hat{p}$. Methods required for calculating integrals of the (35.31) type are discussed, for instance, in [9]. We will give the result of calculating (35.31) that is valid when we assume that $p^2 - m^2 \ll m\lambda$ and ignore terms of the order of m/Λ:

$$\begin{aligned} \overline{M}(p) &= \delta m - [Z_2^{-1} - 1 + R(p)](\hat{p} - m), \\ \delta m &= \frac{3\alpha}{4\pi} m \left(\ln \frac{\Lambda^2}{m^2} + \frac{1}{2}\right), \\ Z_2^{-1} - 1 &= \frac{\alpha}{2\pi}\left[\frac{1}{2} \ln \frac{\Lambda^2}{m^2} + \ln \frac{\lambda^2}{m^2} + \frac{9}{4}\right], \end{aligned} \qquad (35.32)$$

and the function $R(p)$ (whose explicit form we will not give here) vanishes at $p^2 = m^2$ and is independent of Λ.

The electron propagator (35.29) for $p^2 \to m^2$ with allowance made for (35.32) can be written thus:

$$G(p) = \frac{1}{Z_2^{-1}(\hat{p}-m)-\delta m} = \frac{Z_2}{\hat{p}-m_0} + O(\alpha^2), \tag{35.33}$$

since to within terms of the order of α we have the following expression for the constant Z_2 (which follows from (35.32)):

$$Z_2 = 1 - \frac{\alpha}{2\pi}\left[\frac{1}{2}\ln\frac{\Lambda^2}{m^2} + \ln\frac{\lambda^2}{m^2} + \frac{9}{4}\right], \tag{35.34}$$

and $Z_2\delta m \simeq \delta m$ with the same accuracy.

In (35.33) we have introduced the quantity

$$m_0 = m + \delta m, \tag{35.35}$$

which can be called the physical mass of the electron, with δm the electromagnetic mass of the electron. Note that when $\Lambda \to \infty$, δm diverges logarithmically, while in the classical theory the electromagnetic mass diverges linearly [2]. The mass of the "bare" electron, m, and the electromagnetic correction δm cannot be observed separately. Only their sum (35.35), the mass of a real (physical) electron, can be observed.

Thus, when $\hat{p} \to m_0$, we have for the electron propagator

$$G(p) \to \frac{Z_2}{\hat{p}-m_0}. \tag{35.36}$$

Let us use this result to establish a relationship between bispinors $u(p)$ and $U(p)$ (see (35.27)). Since the propagator is quadratic in the bispinor amplitudes (see (22.16) and (22.42)), from (35.36) follows the sought expression for the bispinors:

$$U(p) = \sqrt{Z_2}\,u(p), \quad \bar{U}(p') = \sqrt{Z_2}\,\bar{u}(p'), \tag{35.37}$$

both corresponding to a physical electron. The renormalization of mass has already been carried out, so that $p^2 = p'^2 = m_0^2$, where m_0 is the physical mass given by (35.35).

35.4 Correction to a Vertex

The vertex operator (35.23) determines the correction (35.21d) to the lower-order amplitude (35.21a), which correction is due to the exchange of a virtual photon between the initial and final electrons. Regularization of integral (35.23) is carried out according

to rule (35.30). We will give only the result of such regularization for small momentum transfer, $q = p' - p$ $(-q^2 = \mathbf{q}^2 \ll m^2)$ [12]:

$$\Lambda^\mu = \gamma^\mu (Z_2^{-1} - 1) + \frac{\alpha}{3\pi} \gamma^\mu \frac{q^2}{m^2} \left(\ln \frac{m}{\lambda} - \frac{3}{8} \right) + \frac{\alpha}{8\pi m} (\hat{q}\gamma^\mu - \gamma^\mu \hat{q}). \quad (35.38)$$

Now let us add the amplitude (35.26), which can be written in the form (if we allow for (35.37))

$$A_m = -eZ_2 \bar{u}(p') \gamma^\mu a_\mu u(p), \quad (35.26a)$$

to the amplitude (35.21d), into which we must substitute (35.38). Noting that, by virtue of (35.32) and (35.34),

$$Z_2 + (Z_2^{-1} - 1) = 1,$$

we obtain

$$\bar{A}_m = A_m + A_{(d)}^{(3)}$$
$$= -e\bar{u}(p') \left[\gamma^\mu + \gamma^\mu \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\ln \frac{m}{\lambda} - \frac{3}{8} \right) + i \frac{\alpha}{4\pi m} \sigma^{\mu\nu} q_\nu \right] \bar{u}(p) a_\mu(q), \quad (35.39)$$

which amplitude does not depend on the cutoff parameter Λ but contains the photon "mass" λ. Here

$$\sigma^{\mu\nu} \equiv \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu).$$

35.5 Vacuum Polarization. Renormalization of Charge

Let us now consider the amplitude (35.21e). In Figure 35.1 this amplitude is represented by diagram (e) containing an electron loop. The amplitude describes what is known as vacuum polarization, that is, the polarization of the vacuum state of the electron-positron field when an external field is switched on. Diagram (e) can be given the following graphic interpretation: the external field creates a virtual electron-positron pair that is the source of an additional electromagnetic field δa_μ (35.24).

The additional term δa_μ is expressed in terms of the polarization operator $P^{\mu\nu}(q)$ defined in (35.25). The integral in (35.25) contains a strong (quadratic) divergence. Its regularization is carried out by subtracting from (35.25) a similar integral obtained from the first by replacing m with parameter $M \gg m$:

$$\bar{P}^{\mu\nu}(q) = P^{\mu\nu}(q, m^2) - P^{\mu\nu}(q, M^2). \quad (35.40)$$

Note that the polarization operator has, obviously, the following structure (see (35.25)):

$$P_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) P_{\text{tr}}(q^2) + \frac{q_\mu q_\nu}{q^2} P_{\text{lon}}(q^2),$$

where the first term on the right-hand side is known as the transverse part (its contraction with q^μ or q^ν yields zero) and the second as the longitudinal part. Only the transverse part has physical meaning because it is this part that ensures that the matrix elements of the physical processes are gauge invariant (e.g. (35.24) and (35.21e)). Regularization of $P^{\mu\nu}$ by the rule (35.40) immediately gives the transverse tensor

$$\overline{P}_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) P(q^2). \tag{35.41}$$

Here is the result of calculation of (35.41) for $|q^2|/m^2 \ll 1$ [12]:

$$P(q^2) = -\frac{\alpha}{\pi} q^2 \left[\frac{1}{3} \ln \frac{M^2}{m^2} + \frac{1}{15} \frac{q^2}{m^2}\right]. \tag{35.42}$$

Substituting (35.41) and (35.42) into (35.24), we get the correction to the potential of the external electromagnetic field caused by polarization of the vacuum of the electron-positron field:

$$\delta a_\mu(q) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \frac{\alpha}{\pi} \left[\frac{1}{3} \ln \frac{M^2}{m^2} + \frac{1}{15} \frac{q^2}{m^2}\right] a^\nu(q). \tag{35.43}$$

We substitute (35.43) into amplitude (35.21e) and add the result to (35.39). As a result we arrive at an expression for the scattering amplitude that allows for all radiative corrections of the order of e^2 (the diagrams (a) to (e) in Figure 35.1):

$$A_R = A^{(1)} + A^{(3)}_{(b)} + A^{(3)}_{(c)} + A^{(3)}_{(d)} + A^{(3)}_{(e)} = -e\bar{u}(p') \left[\gamma^\mu \left(1 - \frac{\alpha}{3\pi} \ln \frac{M^2}{m^2}\right)\right.$$
$$\left. + \gamma^\mu \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\ln \frac{m}{\lambda} - \frac{23}{40}\right) + i \frac{\alpha}{4\pi m} \sigma^{\mu\nu} q_\nu\right] a_\mu(q)\, u(p), \tag{35.44}$$

which amplitude depends on the cutoff parameter $M \gg m$. If we send q^2 to zero, (35.44) transforms, in the case of the Coulomb potential (35.16), into

$$A_R = -e^2 \left(1 - \frac{\alpha}{3\pi} \ln \frac{M^2}{m^2}\right) \frac{4\pi Z}{\mathbf{q}^2} \bar{u}(p') \gamma^0 u(p), \tag{35.45}$$

that is, differs from the lower-order amplitude (35.15) by a factor of

$$Z_3 = 1 - \frac{\alpha}{3\pi} \ln \frac{M^2}{m^2}. \tag{35.46}$$

Now let us renormalize the electron charge by postulating that the observed (physical) charge e_0 is a quantity connected with the "bare" electron charge e through the relation

$$e_0^2 = Z_3 e^2, \tag{35.47}$$

where Z_3 is the renormalization constant given by (35.46). For the sake of clarity we can write this relation as

$$e_0^2 = e^2 + \delta e^2, \quad \delta e^2 = -\frac{\alpha}{3\pi} \ln \frac{M^2}{m^2}, \tag{35.47a}$$

and interpret the additional negative term δe^2 as the result of screening of the "bare" charge by the charges of the virtual electron-positron pairs.

Upon charge renormalization the amplitude (35.44) assumes the form

$$A_R = -e_0 \bar{u}(p') \left\{ \gamma^\mu \left[1 + \frac{\alpha}{3\pi} \frac{q^2}{m^2} \left(\ln \frac{m}{\lambda} - \frac{23}{40} \right) \right] + i \frac{\alpha}{4\pi m} \sigma^{\mu\nu} q_\nu \right\} u(p)\, a_\mu(q), \tag{35.48}$$

where e_0 and m stand for the observed charge and mass of a real electron. The amplitude is independent of the cutoff parameters Λ and M but still contains the "photon mass" λ.

Note that upon charge renormalization there appears a finite additional term δa_μ^R in the external field, and this term can actually be observed. For $|q^2|/m^2 \ll 1$ this term, in accord with (35.43), is

$$\delta a_\mu^R(q) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{\alpha}{15\pi} \frac{q^2}{m^2} a^\nu(q).$$

This term contributes to the scattering cross section (see (35.48)) and the shift in the energy levels of the bound states of an electron in an external field (e.g. the shift in the hydrogen atom levels known as the Lamb shift, with the vacuum fluctuations of the electromagnetic field providing the main contribution to this shift) [3, 12, 13].

The polarization operator $P_{\mu\nu}(k)$ considered above also emerges when we have to calculate the radiative corrections to processes involving real and virtual photons. For instance, allowing for corrections of the order of e^2 to the photon propagator $D_{\mu\nu}(k)$ (see Figure 35.4) results in replacing it with

$$G_{\mu\nu}^{(\gamma)}(k) = D_{\mu\nu}(k) + D_{\mu\alpha}(k) \frac{P^{\alpha\beta}(k)}{4\pi} D_{\beta\nu}(k),$$

and as $k^2 \to 0$ we get (with the accepted gauge of $D_{\mu\nu}$)

$$G_{\mu\nu}^{(\gamma)} \to \frac{4\pi Z_3}{k^2} g_{\mu\nu},$$

where Z_3 is the renormalization constant (35.46).

If there are real photons involved in the reaction, then allowing for corrections to the appropriate external lines leads to the multi-

plication of the "bare" photon amplitude by $\sqrt{Z_3}$ (cf. the derivation of (35.37)). However, to a vertex with an incoming photon line there corresponds a "bare" charge e, whereby in the matrix element there appears a factor $\sqrt{Z_3}e = e_0$, the renormalized charge. This means

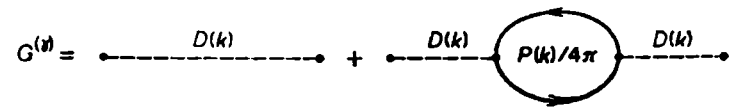

Figure 35.4

that upon renormalization of charge there is no need to allow for corrections to the external photon lines. In what follows we denote the renormalized charge by e.

35.6 Anomalous Magnetic Moment of the Electron

Let us discuss the physical effects that emerge from the last term in amplitude (35.48). If we compare (35.48) with (35.38) and (35.39), we will see that this term is due solely to the correction to the vertex, Λ^μ (diagram (d) in Figure 35.1). In the limit $q \to 0$ only the terms linear in q are retained in (35.48). The result is

$$A_R \simeq -e\bar{u}(p')\left(\gamma^\mu + i\frac{\alpha}{2\pi}\frac{\sigma^{\mu\nu}q_\nu}{2m}\right)u(p)\,a_\mu(q). \qquad (35.49)$$

We transform this amplitude using the relationship

$$2m(\bar{u}'\gamma^\mu u) = (\bar{u}'u)(p'^\mu + p^\mu) + i(\bar{u}'\sigma^{\mu\nu}u)\,q_\nu, \qquad (35.50)$$

which follows from the Dirac equations $(\hat{p} - m)\,u = 0$ and $\bar{u}'(\hat{p}' - m) = 0$ and the commutation rules for the γ^μ-matrices. The result is

$$A_R = -e\bar{u}'\left[\frac{(p'+p)^\mu}{2m} + i\left(1+\frac{\alpha}{2\pi}\right)\frac{\sigma^{\mu\nu}q_\nu}{2m}\right]u a_\mu. \qquad (35.51)$$

As we see, the spin effects in the electron interactions determine the second term, while the first corresponds to a spinless particle. For greater clarity, we introduce the Fourier transform of the tensor of the external field,

$$F_{\mu\nu}(q) = -i(q_\mu a_\nu(q) - q_\nu a_\mu(q)),$$

and rewrite the spin addition to (35.51) in the form

$$\delta_m A_R = \frac{1}{2}\frac{e}{2m}\left(1+\frac{\alpha}{2\pi}\right)\bar{u}'(\sigma^{\mu\nu}F_{\mu\nu})\,u. \qquad (35.52)$$

Expressing $\sigma^{\mu\nu}F_{\mu\nu}$ in terms of Σ and α and the field strengths $\mathbf{E}$ and $\mathbf{H}$ via (8.55), (8.56), and (10.11a), we arrive at (35.52) in three-dimensional form:

$$\delta_m A_R = -\mu\bar{u}'\,(\Sigma\cdot\mathbf{H} - i\boldsymbol{\alpha}\cdot\mathbf{E})\,u, \tag{35.53}$$

where

$$\mu = \mu_0\,(1 + \alpha/2\pi), \tag{35.54}$$

with $\mu_0 = e\hbar/2mc$ the Bohr magneton.

Formula (35.53) describes, obviously, the interaction of an external magnetic field with the electron, whose magnetic moment μ differs from the common (normal) Dirac moment by the quantity

$$\mu' = \frac{\alpha}{2\pi}\mu_0, \tag{35.54a}$$

which is known as the anomalous magnetic moment. The existence of such a moment has been proved by experiments. The theory enables calculating the next correction (of the order of α^2) to the normal moment μ_0:

$$\mu = \mu_0\left(1 + \frac{\alpha}{2\pi} - 0.328\,\frac{\alpha^2}{\pi^2}\right). \tag{35.55}$$

This formula is also in good agreement with the experimental data (with an accuracy higher than one percent).

35.7 Cross Section of Scattering in an External Field with Radiative Corrections Taken into Account

Let us consider the scattering cross section in which we allow for corrections of the order of α to the cross section $d\sigma^{(1)}$ in first-order perturbation theory (the so-called Born approximation), which cross section is given by Mott's formula (35.20). The scattering amplitude with allowance made for the radiative corrections, $A_R = A^{(1)} + A^{(3)}$, corresponds to the sum of diagrams in Figure 35.1 (the explicit form of the scattering amplitude for small $|q^2| \ll m^2$ is given in (35.44)), but proves to be insufficient. Together with the radiative corrections $A^{(3)} \sim Z\alpha^2$ to the Born amplitude $A^{(1)} \sim Z\alpha$ we must allow for the second Born approximation, with the corresponding amplitude $A^{(2)} \sim (Z\alpha)^2$, and the diagram in Figure 35.5. Indeed, to within terms of the order of α^3, the scattering cross section is given by the following formula:

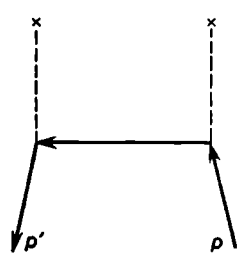

Figure 35.5

$$d\sigma \propto |A^{(1)} + A^{(2)} + A^{(3)}|^2 \simeq |A^{(1)}|^2 + 2\mathrm{Re}\,(A^{(1)}A^{(2)*}) + 2\mathrm{Re}\,(A^{(1)}A^{(3)*})_{\mathrm{R}} \tag{35.56}$$

that is, $A^{(2)}$ and $A^{(3)}$ provide contributions of the same order of magnitude if the nuclear charge Z is of the order of unity.

The scattering cross section obtained in this manner (we call it elastic; see below) can be represented as a sum of three terms,

$$d\sigma_e = d\sigma^{(1)} + d\sigma^{(2)} + d\sigma_R^{\lambda}, \tag{35.57}$$

with the separate terms corresponding to expansion (35.56).

The correction due to the second Born approximation $d\sigma^{(2)}$ is expressed, as shown by calculations, in terms of $d\sigma^{(1)}$ as follows (see [12]):

$$\begin{aligned} d\sigma^{(2)} &= Z\alpha B(\theta)\, d\sigma^{(1)}, \\ B(\theta) &= \pi v \sin\frac{\theta}{2}\,\frac{1-\sin(\theta/2)}{1-v^2\sin^2(\theta/2)}, \end{aligned} \tag{35.58}$$

where $v = |\mathbf{p}|/\varepsilon$ is the electron speed, and $\theta = (\widehat{\mathbf{p}, \mathbf{p}'})$ is the scattering angle.

The radiative correction $d\sigma_R^{\lambda} \propto 2\mathrm{Re}\,(A^{(1)}A^{(3)*})$ has a rather complicated form for arbitrary q^2. For this reason we will not give it here; instead we give its general structure:

$$d\sigma_R^{\lambda} = \alpha\left[R(q^2) - F(q^2)\ln\frac{m}{\lambda}\right] d\sigma^{(1)}. \tag{35.59}$$

This correction contains the term

$$d\sigma_{\lambda} = -\alpha F \ln\frac{m}{\lambda}\, d\sigma^{(1)}, \tag{35.60}$$

which possesses infrared divergence (if the "photon mass" is sent to zero). For small $|q^2| \ll m^2$ the function F, as seen from (35.48), has the simple form

$$F = -\frac{2}{3\pi}\frac{q^2}{m^2}. \tag{35.60a}$$

The dependence of cross section (35.57) on the fictitious "photon mass" λ, which was introduced in the regularization of divergent integrals (see above) has no physical meaning. This problem can be resolved in the following manner. Only the cross section of the process in which the soft-photon bremsstrahlung is allowed for can have any meaning (see Section 25 and Figure 35.6 with the appropriate diagrams); the energy of the photons must be lower than a certain fixed value ω_{max}. This quantity, ω_{max}, is determined by the energy resolution ΔE of the experimental device, which actually registers events corresponding both to elastic scattering and inelastic

scattering accompanied by the emission of soft photons with energies $\omega \leqslant \Delta E = \omega_{max}$. Events of these two types cannot be distinguished experimentally, since the electron energy is measured to within ΔE.

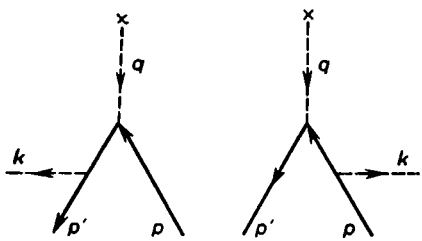

Figure 35.6

If we now calculate via perturbation theory the cross section of scattering accompanied by emission of a soft photon with an energy $\omega \leqslant \omega_{max}$, the result will be (see [12, 13])

$$d\sigma_{emis} = \alpha F \ln \frac{2\omega_{max}}{\lambda} d\sigma^{(1)}, \quad (35.61)$$

where F is the same function as in (35.57), and in calculating the cross section (35.61) we must employ the method of regularization of the total cross section by introducing the "photon mass" λ (in contrast to the method employed in Section 25). The thing is that although observables do not depend on the regularization method, all divergent quantities that occur in intermediate calculations must be regularized by a single method [13]. It is for this reason that the methods for calculating (35.59) and (35.61) coincide. Of course, the method used in Section 25 could have also be employed, but only simultaneously for both quantities.

Adding (35.60) to (35.61), we get

$$d\sigma_{emis} + d\sigma_{\lambda} = -\alpha F \ln \frac{m}{2\omega_{max}} d\sigma^{(1)}, \quad (35.62)$$

that is, the contribution from soft virtual photons ($|\mathbf{k}| \sim \lambda$) cancels out the contribution from soft real photons. It can be shown that this is true for any scattering process.

Replacing in (35.59) the cross section $d\sigma_{\lambda}$ with the sum (35.62), we obtain the following expression for the observed radiative correction:

$$d\sigma_R = \alpha \left[R(q^2) - F(q^2) \ln \frac{m}{2\omega_{max}} \right] d\sigma^{(1)}, \quad (35.63)$$

and the cross section of purely elastic scattering transforms into the cross section of the process that allows for emission of soft photons with energies lower than ω_{max}:

$$d\sigma = d\sigma^{(1)} + d\sigma^{(2)} + d\sigma_R. \quad (35.64)$$

This cross section contains no divergences but depends on parameter ω_{max} which is to be expected by virtue of the definition of a scattering process as a process involving the emission of any number of soft photons with energies lower than ω_{max}, a quantity depending on the concrete experimental setup. As (35.63) shows, the smaller

the value of ω_{max}, the smaller the cross section (35.64). Passing to the limit $\omega_{max} \to 0$ in (35.64) is, of course, impossible, since the formula was obtained in calculating the cross section of emission of soft photons via perturbation theory, which becomes invalid in this limit. If we go beyond the limits of perturbation theory, then, as noted in Section 25, the limit of the cross section as $\omega_{max} \to 0$ is zero, which implies that no purely elastic scattering is possible in this limit.

Note that the factor inside the square brackets in (35.63) is a universal function independent of the type of external field (this field determines only the form of cross section $d\sigma^{(1)}$).

Finally, we give the explicit form of the radiative correction (35.63) in two limiting cases [13]. In the nonrelativistic approximation,

$$d\sigma_R = -\frac{2\alpha}{3\pi}\frac{\mathbf{q}^2}{m^2}\left(\ln\frac{m}{2\omega_{max}}+\frac{19}{30}\right)d\sigma^{(1)}, \quad \mathbf{q}^2/m^2 \ll 1, \quad (35.65)$$

while in the ultrarelativistic limit (the electron energy ε is much higher than m) we have

$$d\sigma_R = -\frac{2\alpha}{\pi}\ln\frac{\mathbf{q}^2}{m^2}\ln\frac{\varepsilon}{\omega_{max}}d\sigma^{(1)}, \quad \mathbf{q}^2/m^2 \gg 1. \quad (35.66)$$

Thus, by using the calculation of the cross section of electron scattering in an external field as an example we have demonstrated that allowing for the higher-order perturbation-theory approximations in quantum electrodynamics requires renormalizing the charge and mass of the electron. After such renormalization has been carried out, Feynman's rules enable finding unique finite expressions for all observable quantities in any order in parameter e^2. All this can be proved rigorously [12, 9].

So far all experiments conducted with the aim of verifying the predictions of quantum electrodynamics have shown remarkable agreement with the predictions.

36. The Basic Properties of Synchrotron Radiation and Their Experimental Verification

Roughly forty years have passed since synchrotron radiation was discovered, but even now the interest toward this physical phenomenon has not flagged; on the contrary, the physical community has evidenced a revival of interest in this topic. Emission of synchrotron radiation was first discovered as a by-product in the motion of electrons in a circular accelerator, synchrotron. Hence, the name of the radiation. The development of the theory of synchrotron radiation, that is, a theory that explained all properties of such ra-

diation, came before the experiment. This emphasized the interest toward the experimental study of the new physical phenomenon. In the 1950s the problem of synchrotron radiation acquired considerable importance in connection with the design and construction of high-energy electron cyclic accelerators. It was found that, at relativistic speeds of the electrons in the magnetic field of the accelerator, synchrotron radiation plays a fundamental role; precisely, it determines the dynamics of the particles in the accelerator: radiative energy losses, the classical radiative damping of betatron oscillations, and the quantum fluctuations of the particle trajectories.

In those years the theory of the radiation generated by relativistic particles moving in a magnetic field attracted the attention of astrophysicists in connection with the problem of explaining the nature of the electromagnetic radiation generated in outer space. In 1950 the Swedes H. Alfvén and N. Herlofson [36.1] proposed a hypothesis that explained the origin of the nonthermal radio-frequency radiation in the Universe by the mechanism of synchrotron radiation generated by fast particles decelerated by magnetic fields. In the USSR two scientists, V. L. Ginzburg [36.2] and I.S. Shklovsky [36.3], pioneered in these studies (1953). Both theory and experiment have developed with due regard for the important role of synchrotron radiation, and this explains the amazing results achieved in radio-astronomy.

Scientists have established that the radio-emission of the supernovae (Cassiopeia A, Taurus A, and the radio galaxies of the M-87 type) and the sporadic radio-emission of the Sun and Jupiter are closely linked with synchrotron radiation. In some cases synchrotron radiation was found not only in the radio-frequency range but also in the optical and X-ray ranges (the Crab Nebula). The discovery of superstars, or quasars, has shown that the strong radio-emission of these extraordinary and extremely far-off objects is, at least partially, of synchrotron-radiation origin. The interest toward synchrotron radiation also intensified due to the discovery of pulsars—radio sources that emit short pulses of radiation at very regular intervals. The high degree of polarization of the radiation emitted by a pulsar suggests that the mechanism of this emission is similar to the one that generates synchrotron radiation. Thus, at present synchrotron radiation in radio-astronomy has firmly gained ground.

In the last decade new interest toward synchrotron radiation has emerged in connection with fundamental studies in spectral regions where only lately have new powerful sources of radiation emerged, such regions as the vacuum ultraviolet and X-rays up to soft gamma rays. The wavelengths in this specific region are of the order of the dimensions of atoms and molecules, in view of which such electromagnetic radiation interacts well with solids. Being a unique source of electromagnetic waves, synchrotron radiation is of

high intensity in a broad frequency range—from the infrared region to hard X-rays. This opens up new possibilities for applying radiation as a means of studying solids, say, in atomic and molecular physics, radiochemistry, and molecular biology.

The present stage of studies involving synchrotron radiation is characterized not only by the work done in applying radiation from known sources but also by the construction of special sources of synchrotron radiation, primarily electron storage rings. The field of application of synchrotron radiation is ever increasing, and this opens up new regions before the experimentors, regions that not so long ago were firmly closed for investigation.

Thus, the problem of emission of electromagnetic waves by electrons moving in a magnetic field has acquired a fundamental aspect, although at first glance it seems to be a purely electrodynamical problem.

36.1 Angular Spectral Characteristics of Synchrotron Radiation

To study the angular distribution of the power carried by synchrotron radiation from a qualitative point of view, we recall that (as shown in Section 16) the radiation of a nonrelativistic electron has a markedly dipole nature, whereby the spatial distribution of the power carried by synchrotron radiation looks like a torus (Figure 36.1).

Let us assume that ψ' is the angle between the direction of radiation propagation and the velocity of the particle in the reference frame in which the observer is at rest. Then the angle ψ at which the radiation is observed in the laboratory system can be found from the law of velocity addition (the aberration formula)

$$\sin \psi = \frac{\sqrt{1-\beta^2}\sin \psi'}{1+\beta \cos \psi'}. \tag{36.1}$$

Assuming also that $\psi' = \pi/2$, the power maximum associated with dipole radiation, we obtain

$$\sin \psi \simeq \delta\psi \simeq \sqrt{1-\beta^2} = \frac{mc^2}{E} = \frac{1}{\gamma}. \tag{36.2}$$

Thus, because of the Doppler effect, synchrotron radiation for an electron moving at a relativistic speed is concentrated within a narrow cone about the instantaneous direction of electron motion and is directed forward. The angle at the cone's vertex $\delta\psi = mc^2/E$ decreases rapidly as the energy grows. This corresponds to the result obtained in Section 17 by a somewhat different method.

Let us now dwell on the features of the spectral composition of synchrotron radiation from the qualitative standpoint. Consider

the synchrotron radiation that arrives at point P (Figure 36.2). Due to the characteristic angular distribution of the radiated power, the

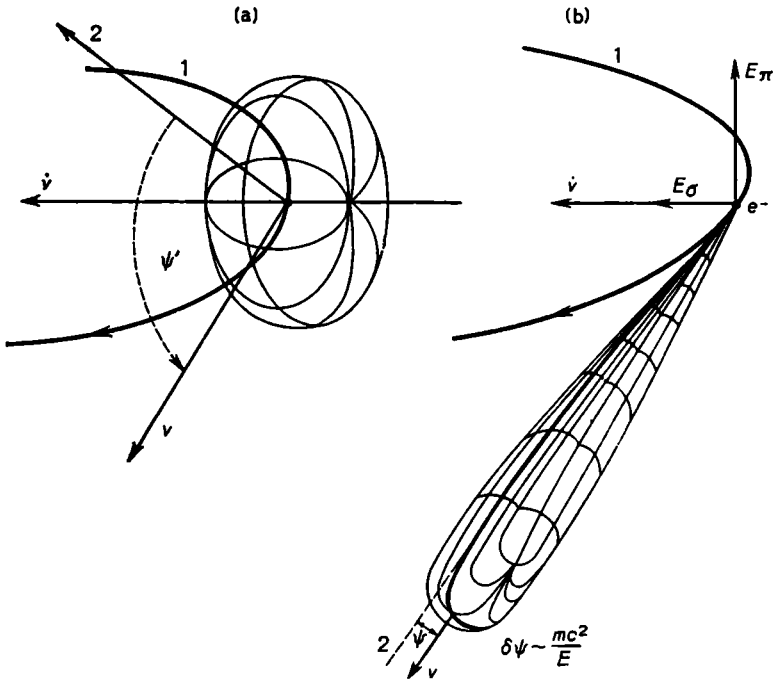

Figure 36.1 The angular distribution of synchrotron radiation for (*a*) low-energy electrons ($v \ll c$), and (*b*) ultrarelativistic electrons ($v \sim c$).

observer sees the electron moving over a short path of its circular trajectory. If $l_r = R\delta\psi$ is the effective arc length over which the

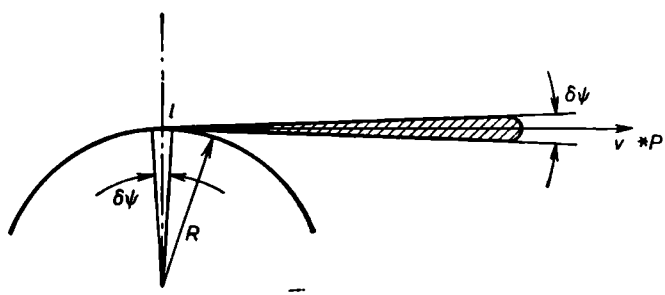

Figure 36.2

radiation is formed (see (17.20)), then the time it takes the electron to travel this path is

$$\tau' = l/c. \tag{36.3}$$

The radiation pulse (its duration is denoted by Δt) in the laboratory system will be very short (even allowing for the retardation in the electromagnetic wave), like a flash, that is,

$$\Delta t \sim \gamma^{-2}\tau', \tag{36.4}$$

in accord with the results obtained in Section 17. All this means that at point P the observer sees not a single wave but a wave packet, since a single monochromatic wave cannot carry a signal limited in time. As is known, the intervals Δt and $\Delta\omega$, which characterize the duration of the signal and the frequency composition of the spectrum, are related thus:

$$\Delta t \Delta\omega \simeq 1. \tag{36.5}$$

Thus, the observer at point P will see a whole set of harmonics of the spectrum, including frequencies of the order of $\Delta\omega \sim 1/\Delta t$. Obviously, there can be two limiting cases here.

(a) *Synchrotron radiation.* In this case the length of the arc over which the radiation is formed is $l_\tau = R\delta\psi = R/\gamma$. This implies that the observed spectrum has a maximum in the higher-harmonics region for the revolution frequency $\omega_0 = c/R$:

$$\Delta\omega \sim \frac{1}{\Delta t} \sim \frac{\gamma^2}{\tau'} = \omega_0\gamma^3 = \omega_0\left(\frac{E}{mc^2}\right)^3. \tag{36.6}$$

Thus, the observer will detect not only the radiation with the fundamental frequency of revolution but also the higher harmonics, which may, depending on the energy of the particles, lie in the visible spectrum (the "luminous" electron) or even in the vacuum ultraviolet or X-ray parts of the spectrum. As shown in Section 17, the characteristic curve $f(y)$ of the distribution of the radiated power of synchrotron radiation has, depending on the number of the radiated harmonic $\nu = \omega/\omega_0$,

$$y = \frac{2}{3}\frac{\omega}{\omega_c} = \frac{2}{3}\nu\frac{\omega_0}{\omega_c} = \nu\frac{2}{3}\left(\frac{mc^2}{E}\right)^3, \tag{36.7}$$

a maximum corresponding to the value of the critical frequency $\omega_c = \omega_{cr}^{syn} = \omega_0 (E/mc^2)^3$. The spectral formula (17.52) was first derived by A.A. Sokolov and D. D. Ivanenko [36.4] (1948) and later by J. Schwinger [36.5] (1949). The f versus y graph is shown in Figure 17.1.

Note that the curve representing the spectral distribution of the power carried by synchrotron radiation in Figure 17.1 resembles the famous Planck curve for blackbody radiation (e.g. see [36.6]). Comparing the radiation maxima, we can see that a synchrotron with 5-GeV electrons resembles a black body with a temperature of 10^7 K. Other man-made sources of such radiation are high-temperature plasmas and nuclear explosions.

(b) *Undulator radiation.* Let us take up the second limiting case of the radiation generated by relativistic particles. Suppose that now the electron is moving in a magnetic field with a period l_0 (an undulator) (Figure 36.3). If the length of the arc over which the radiation is formed is much smaller than l_0 ($l_r = R\delta\psi \ll l_0$), emission will be determined by the local curvature of the trajectory, and we again arrive at a spectrum characteristic of synchrotron radiation.

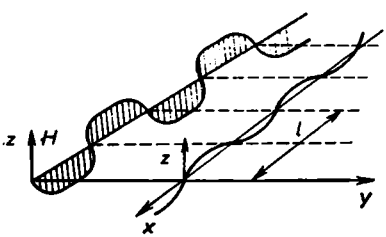

Figure 36.3 The motion of a particle in an undulator.

Another case is possible, however: the length of the arc over which the radiation is formed can be of the order of the field period l_0 (the undulator mode). Then from (36.6) we obtain

$$\Delta\omega^{\text{und}} \sim \frac{c\gamma^2}{l_0} = \frac{c}{l_0}\left(\frac{E}{mc^2}\right)^2. \quad (36.8)$$

The radiation is still synchrotron: all radiation is directed forward (in the direction of electron motion), but the power maximum is now at a somewhat different value of the critical frequency. The radiation is emitted from the entire electron trajectory.

As the theory shows (e.g. see [36.7, 36.8]), if the electron in the undulator emits radiation in the forward direction, the frequency of this radiation is given by the formula

$$\omega = n\frac{2\pi c}{l_0}\left(\frac{E}{mc^2}\right)^2, \quad (36.9)$$

where $n = 1, 2, 3, \ldots$ is the number of the harmonic. This implies that the power maximum in undulator radiation falls on the fundamental tone $n = 1$, since

$$\omega_{\text{cr}}^{\text{und}} = \frac{2\pi c}{l_0}(E/m_0c^2)^2. \quad (36.10)$$

In this respect the emission spectrum of the electron in the undulator differs considerably from the spectrum of synchrotron radiation.

Now let us investigate the problems involved in observing the angular spectral distribution of the radiated power of synchrotron radiation in experiments. Synchrotron radiation was first predicted theoretically by D. D. Ivanenko and I. Ya. Pomeranchuk [36.9] in 1944. The first attempts to observe this phenomenon in experiments were not very successful. Indirectly (by the radiative shrinking of the radius of the electron orbit) synchrotron radiation was observed by J. Blewett [36.10] in 1946 on an 80-MeV synchrotron. However, attempts to observe synchrotron radiation directly were unsuccessful since Blewett estimated the spectral composition of synchrotron radiation incorrectly and looked for it in the microwave region, as-

suming that the intensity maximum falls on the fundamental frequency of electron revolution in the magnetic field.

After the papers by L. A. Artsimovich and I. Ya. Pomeranchuk [36.11] (1946) and J. Schwinger [36.12] (1946) were published, in which they estimated the critical harmonic corresponding to the emission maximum, it became clear that synchrotron radiation must be sought for in the visible spectrum. In 1947, using the same 80-MeV synchrotron, a group headed by H. C. Pollock [36.13] observed this radiation in the form of a light glow in the vacuum chamber.

Further investigations in the classical theory of synchrotron radiation were directed at studying the spectral and angular composition of the radiation. D. D. Ivanenko and A. A. Sokolov [36.4] (1948) were able, using approximate methods of quasi-classical quantum mechanics, to obtain a closed asymptotic formula for describing the entire spectrum of synchrotron radiation (see (17.52) and Figure 17.1).

The theoretical results thus obtained enabled making a detailed quantitative study of the various laws governing the radiation. Pollock's group [36.14] (1948) experimentally studied the distribution of the radiated power within a wavelength range from 3500 to 7000 Å, which corresponds to 30- to 80-MeV electrons. The radiation was found to be concentrated within a narrow cone in the form of a dark red spot at 30 MeV and a bright blueish-white spot at 90 MeV. The glow proved to be so bright that it was seen even in daylight. Yu. M. Ado and P. A. Cherenkov [36.15] (1956), using the FIAN synchrotron facilities, studied the visible part of the electron emission spectrum at 150 to 225 MeV.

Good agreement with the theoretical predictions was found in all experiments, at least in the visible spectrum. Synchrotron radiation became the most direct factor in detecting electrons: it is emitted directly by electrons moving in an electromagnetic field, a medium without microstructure. The electron becomes literally "luminous" (D. D. Ivanenko and A. A. Sokolov [36.4]).

The development of the accelerator techniques made it possible to move up the energy scale, so to say. In the period from 1953 to 1956, D. Tomboulian and P. Hartman [36.16], using the synchrotron facilities at Cornell University (320 MeV), carried out systematic investigations of the spectral composition of synchrotron radiation in the vacuum ultraviolet region (400 to 60 Å). The general setup was the same as in the experiments on the spectral composition of synchrotron radiation in the visible spectrum. Spectrum analysis was performed on the synchrotron radiation, and the various components were analyzed radiometrically, photographically, and by other means.

Experiments in the region of vacuum ultraviolet also showed good

agreement with the theoretical results. This became extremely important also in connection with the discussion on the invalidity of the classical theory of synchrotron radiation in the 200 to 250-MeV region [36.17]. Spectrum studies and direct measurements of total energy losses carried out by D. R. Corson [36.18] on the same 320-MeV synchrotron revealed no divergences from the classical description.

It is important to note this fact because even now, when the criterion for the validity of the classical theory of radiation up to energies $E \sim E_{1/2} = mc^2 (mcR/\hbar)^{1/2}$ has been firmly established [36.19], there are attempts (e.g. see [36.20]) to revise this theory with the aim of introducing quantum effects into the radiation intensity at energies that are much lower that $E_{1/2}$.

In 1966, G. Bathov, E. Freytag, and R. Haensel [36.21] carried out a series of experiments on the DESY synchrotron (Hamburg, FRG) involving the analysis of the synchrotron-radiation spectrum in the X-ray region at electron energies ranging from 4.0 to 6.3 GeV. Even in this range the experimental data was in good agreement with the classical theory.

Experiments involving electron beams (10^{10} to 10^{11} particles in a single acceleration cycle) have shown good agreement with the theory of radiation emitted by a single electron. Thus, synchrotron radiation occurs basically noncoherently, at least in the high-frequency region, which is the most important region for relativistic electrons.

Lately attention has been focused on the problem of generation of electromagnetic waves by electrons moving in a periodic electromagnetic field. A system with a periodic field became known as an "undulator", or "wiggler". Undulator radiation emerges when a fast charge moves along a periodically curved trajectory (see Figure 36.3), on separate sections of which the particle emits radiation due to centripetal acceleration.

The first to point out the possibility of generation of radiation in an undulator was V. L. Ginzburg [36.23] (1947), and in 1951 H. Motz [36.24] carried out the first experiment in the submillimeter range. The initial hopes of creating bunches of electrons that would emit radiation coherently and thus amplify the radiated power faded, and the interest toward undulator radiation slackened somewhat.

An important idea was introduced by R. P. Godwin [36.25] in 1969. He proposed placing the undulator inside the chamber of the electron synchrotron or storage ring (it is common practice to place the undulator in a straight section). This enabled using the synchrotron and undulator radiations simultaneously, since they are formed over different parts of the electron trajectory (Figure 36.4). In view of the specific features of undulator radiation (the maximum falls on the fundamental tone; see (36.9)), it became possible, by selecting the parameters of the setup, to achieve harder radiation than in a synchrotron of the same electron energy (see the paper by

D. F. Alferov, Yu. A. Bashmakov, and E. G. Bessonov [36.8] and also the paper by I. M. Ternov, V. R. Khalilov, V. G. Bagrov, and M. M. Nikitin [36.26]).

Not so long ago, at the Physical Institute of the USSR Academy of Science (the "Pakhra" 1.3-GeV synchrotron) and at the Tomsk Polytechnical Institute (the "Sirius" 1.2-GeV synchrotron), the undulator radiation was studied in experiments. First the radiation was observed as emerging from a flat undulator placed over the orbital plane of the electrons in one of the straight sections of the "Pakhra" synchrotron. A beam of 100-MeV electrons was introduced into the undulator, which consisted of 20 periodicity elements and whose field period was 4 cm. The researchers studied the spectral distribution of the intensity of radiation from the undulator in the 2300-4500 Å wavelength range and found the results to be in good agreement with the theoretical predictions (D. F. Alferov, Yu. A. Bashmakov, K. A. Belovintsev, E. G. Bessonov, and P. A. Cherenkov [36.27] (1977)).

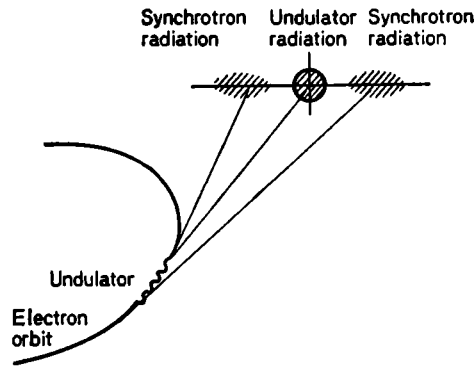

Figure 36.4 The schematic of an experiment in which undulator radiation is detected. With the undulator switched off the central spot (undulator radiation) is absent, and only the synchrotron radiation due to bending magnets is present.

About the same time but with greater detail experiments were conducted in which the "Sirius" electron synchrotron was employed to study the properties of undulator radiation. In one of the straight sections of the synchrotron was placed an iron-free magnetic undulator with a period of 14 cm and 5 periodicity elements. The optimal choice of the field period and the number of periods was dictated by the experimental conditions in the optical range. The spectra of the integral (over the angles) distribution of the radiation on the first, second, and third harmonics were measured for the first time, and so was the polarization of the radiation [36.29]. The experimental data was found to support fairly well the theoretical results (A. N. Didenko, A. V. Kozhevnikov, A. F. Medvedev, and M. M. Nikitin [36.28] (1978)).

At present scientists in many countries (USSR, USA, Italy, France) are building undulators at laboratories with access to accelerators and/or electron storage rings.

36.2 Polarization Properties of Synchrotron Radiation

The reader already knows from Section 17 that synchrotron radiation possesses a strong linear polarization, that is, the electric-field vector in the radiation is directed mainly along the radius of the electron orbit (see Figure 36.1). The Schott formula, which characterizes the angular spectral distribution of radiation's intensity, takes on the following generalized form if we include polarization effects (A. A. Sokolov and I. M. Ternov [36.30] (1956); see (17.14)):

$$\frac{dW}{d\Omega} = \frac{e_0^2 c^2 \beta^2}{2\pi R^2} \nu^2 \{l_2 \beta J'_\nu(\nu\beta \sin\theta) - l_3 \cot\theta J_\nu(\nu\beta \sin\theta)\}^2. \quad (36.11)$$

Assuming that here $l_2 = 1$ and $l_3 = 1$, we get the σ-component of the linear polarization, and assuming that $l_2 = 0$ and $l_3 = 1$, we get the π-component. When calculating the power of radiation that is circularly polarized, we must put $l_2 = \mp l_3 = 2^{-1/2}$ for right-handed and left-handed circular polarizations, respectively.

From the viewpoint of observing synchrotron radiation, the most interesting case is the one in which ultrarelativistic electrons emit the radiation ($1 - \beta^2 \ll 1$). After integrating over the spectrum, we arrive at the following characteristic angular distribution of the linear components of the polarization:

$$\frac{dW_i}{d\Omega} = \frac{e_0^2 c}{32\pi R^2 (1-\beta^2)^{5/2}} f_i(\tau), \quad \tau = \frac{\cos\theta}{\sqrt{1-\beta^2}} \simeq \left(\frac{\pi}{2} - \theta\right)\gamma, \quad (36.12)$$

with

$$f_\sigma = \frac{7}{(1+\tau^2)^{5/2}}, \quad f_\pi = \frac{5\tau^2}{(1+\tau^2)^{7/2}}. \quad (36.13)$$

The graph of these functions is shown in Figure 36.5. Thus, we see that in the orbital plane of the electrons, where $\theta = \pi/2$ and $\tau = 0$, the radiation is purely linearly polarized since the π-component vanishes: $dW_\pi = 0$.

For the total radiated power of synchrotron radiation we can obtain the following formula (see (17.42)):

$$W_\sigma = \frac{7}{8} W, \quad W_\pi = \frac{1}{8} W. \quad (36.14)$$

The first to study the angular-polarization characteristics of synchrotron radiation experimentally were F. A. Korolev and O. F. Kulikov [36.31] (1956). Using the 280-MeV FIAN synchrotron facilities, they were able to photograph both components of the linear polarization and study the angular distribution of the radiated power up to electron energies of 250 MeV.

In Figures 36.6 and 36.7 we compare the experimental data with the theoretical predictions; we also show the photographs of the σ-

and the π-component in the synchrotron radiation. This series of experiments was repeated by F. A. Korolev and O. F. Kulikov on the 680-MeV FIAN synchrotron and by P. Joos [36.32] (1960) on the 320-MeV synchrotron at Cornell University. All these experiments were conducted in the visible range.

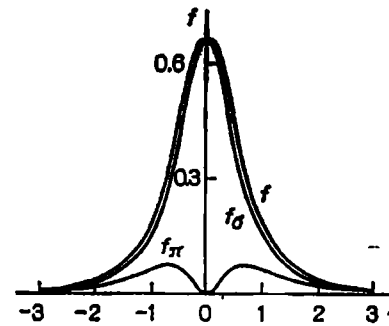

Figure 36.5 Linear polarization versus emission angle.

F. A. Korolev and O. F. Kulikov [36.33] (1960), in their experiments conducted on the 680-MeV FIAN synchrotron, were the first to study the circular polarization of synchrotron radiation (see also [36.32]). The general theory of radiation [36.30] implies that there is a phase shift between the σ- and the π-component of linear polarization equal to $\varphi_\sigma - \varphi_\pi = \delta_0$. For $0 < \theta < \pi/2$ this shift is equal to $-\pi/2$, while for $\pi/2 < \theta < \pi$ this shift is $\pi/2$. In other words, in the first case synchrotron radiation is left-handed elliptically polarized, while in the second case it is right-handed elliptically polarized. Experiments have shown good agreement with the theoretical predictions.

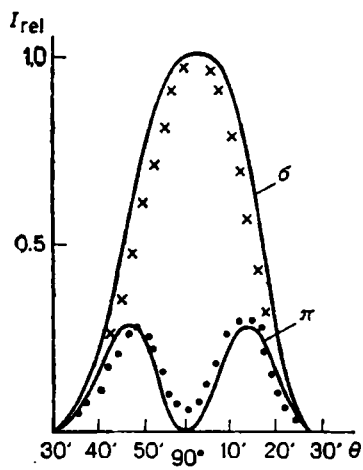

Figure 36.6

In 1968, G. Rosenbaum, B. Feuerbacher, R. Godwin, and M. Skibowsky [36.34] conducted a series of experiments in which they measured the polarization of synchrotron radiation in the vacuum ultraviolet region, while G. Bathov, E. Freytag, and R. Haensel [36.21], using the same synchrotron, measured the polarization components of synchrotron radiation in the X-ray region. Measuring polarization in the ultraviolet region required employing a special device, a vacuum ultraviolet monochromator, while experiments in the X-ray region were carried out via the Compton-scattering method. Studies conducted in the hard region of electron energies and practically over the entire spectrum have shown good agreement with the theory of synchrotron radiation.

The polarization of synchrotron radiation constituted an important aspect in the solution of the riddle of the origin of the radiation arriving from the Crab Nebula. The Crab Nebula, which is an

expanding cloud of gas ejected by a star seen to explode in July 1054 A. D. (such stars are known as supernovae), can be observed at present in the Taurus constellation. The Crab Nebula is still expanding and emitting light; in fact, it emits a lot of light and can be observed by optical methods.

In the 1940s it was established that the Crab Nebula is simultaneously a source of powerful radio-emission in the 1-cm to 10-m range. I. S. Shklovsky [36.35] (1953) put forward a hypothesis according to

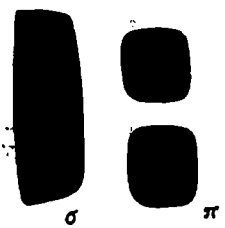

Figure 36.7 Photographs of the σ- and the π-component of linear polarization of synchrotron radiation.

which the radio-emission and the principal part of the optical emission from the Crab Nebula can be explained by synchrotron radiation. Indeed, the nebula contains a relatively high number of relativistic electrons, that is, electrons with energies ranging from 10^{11} to 10^{12} eV. In this way, synchrotron radiation may encompass not only the radio-band but also the optical band.

The polarization properties of synchrotron radiation have played an important role in establishing the nature of the radiation arriving from the Crab Nebula. A series of observations that permitted discovering the polarization of synchrotron radiation was carried out in the optical band by V. A. Dombrovskii and M. N. Vashakidze [36.36] (1954) and in the radio-band by A. D. Kuz'min and V. A. Udal'tsov [36.37] (1957). These experiments completely corroborated the hypothesis on the synchrotron origin of the radiation emitted by the Crab Nebula. Similar experiments [36.38] established the synchrotron origin of the radiation emitted by the M-87 radio galaxy (1956).

36.3 Quantum Effects in Synchrotron Radiation

The question of quantum effects manifesting themselves in synchrotron radiation proved to be nontrivial and interesting. The most obvious criterion for the validity of the classical theory is the one proposed by V. V. Vladimirskii [36.39] and J. Schwinger [36.40], according to which quantum effects can be ignored when the energy of an emitted photon, $\hbar\omega \sim (\hbar c/R)\,(E/mc^2)^3$, is much lower than the electron energy E ($\hbar\omega \ll E$). From this it follows that the clas-

sical approximation is valid if

$$\hbar\omega \ll E, \quad \text{or} \quad E \ll E_{1/2} = mc^2\left(\frac{mcR}{\hbar}\right)^{1/2} = \frac{H_0}{H}mc^2. \qquad (36.15)$$

An estimate of the critical value of energy $E_{1/2}$ shows that this value lies outside the limits that can be achieved on the current crop of accelerators, 10^{13} eV.

The first quantitative calculation of the power carried by synchrotron radiation that allowed for quantum corrections was undertaken by A. A. Sokolov, N. P. Klepikov, and I. M. Ternov [36.41] (1952), who confirmed the above reasoning concerning the limits of applicability of the classical theory of radiation (see Section 29):

$$W^{\text{quan}} = W^{\text{cl}}\left\{1 - \frac{55\sqrt{3}}{16}\frac{\hbar}{mcR}\left(\frac{E}{mc^2}\right)^2 + \ldots\right\}$$

$$= W^{\text{cl}}\left\{1 - \frac{55\sqrt{3}}{16}\left(\frac{E}{E_{1/2}}\right)^2 + \ldots\right\}. \qquad (36.16)$$

Later the same result was obtained by J. Schwinger [36.42] (1954). However, subsequent development of the quantum theory of synchrotron radiation revealed that the quantum fluctuation nature of the radiation should have a strong effect on the trajectory of the particles (A. A. Sokolov and I. M. Ternov [36.43] (1953)). The value of the critical energy $E_{1/5}$, which characterizes the effect that quantum fluctuations have on the trajectory when the electron moving along this trajectory emits a photon,

$$E_{1/5} = mc^2\left(\frac{mcR}{\hbar}\right)^{1/5}, \qquad (36.17)$$

constituted several hundred MeV's which in those years lay within the energy of accelerators being designed. It was predicted that starting from $E_{1/5}$ quantum fluctuations begin to affect the radius of the electron orbit. An electron in this case moves like a Brownian particle and recoils as a result of photon emission. Due to quantum fluctuations of the radiation, the square fluctuation of the radius must grow according to the law

$$\frac{d}{dt}\overline{\Delta R^2} = \frac{d}{dt}(\overline{R^2} - \bar{R}^2) = \frac{55}{48\sqrt{3}}\frac{e^2\hbar}{m^2c^2R}\left(\frac{E}{mc^2}\right)^5. \qquad (36.18)$$

The dependence $\overline{\Delta R^2} = \overline{R^2} - \bar{R}^2 \simeq At$ is characteristic of Brownian motion, as is well known.

Thus, we see that the discrete nature of synchrotron radiation, having no effect on the radiated power, begins to manifest itself at electron energies of the order of several hundred MeV's. Physically the discreteness of the radiation can be explained by the ejection of extremely powerful quanta. Indeed, if we estimate the number of

photons emitted during one revolution of the electron, we can easily find that

$$N \sim \frac{W^{\text{rev}}}{\hbar\omega_{\max}} \sim \frac{e^2}{\hbar c}\frac{E}{mc^2}, \tag{36.19}$$

since $\omega_{\max} \sim \omega_0 (E/mc^2)^3$. This suggests that at energies of the order of several hundred MeV's there can be only several emitted powerful quanta, which emphasizes the discreteness of the radiation.

Thus, there appears a peculiar situation, namely, the oscillations of the electron about its radius of orbit, while being macroscopic in magnitude, obey the laws of quantum mechanics: the radial coordinate of the particle can be determined only with a certain probability. Such motion of the electron can naturally be called a "macroatom". In this connection it is important to emphasize that the effect of quantum fluctuations of the particle trajectory manifests itself in macroscopic motion and constitutes an interesting example from macroscopic quantum theory.

Soon after the discovery of the effect of quantum fluctuations of the trajectory of an electron moving in a magnetic field it became obvious that this effect is of great practical importance from the standpoint of the theory and practice of constructing cyclic electron accelerators.

As is well known, in a cyclic resonance accelerator, or a synchrotron, the electrons move in the focusing magnetic field along a circle that, on the average, has a constant radius. The focusing forces ensure the stability of motion of the particles: when an electron is deflected from the equilibrium orbit, it performs fast so-called betatron oscillations, radial (about the constant radius of the orbit) as well as vertical (about the orbital plane). The electron acquires an acceleration due to the action of an external high-frequency electric field directed along the tangent to the trajectory of the particle. In this case the phase stability of the beam is also ensured: the electrons perform slow phase oscillations about the equilibrium phase.

The effect of the radiation becomes highly important from the viewpoint of the stability of particle motion. Earlier studies, which were carried out via methods of classical theory, demonstrated that the reaction of the radiation leads to radiative damping of the phase oscillations. The damping factor Γ_{ph} proves to be linked with the electron energy E and the radiated power W through the following formula:

$$\Gamma_{\text{ph}} = \frac{3-4n}{1-n}\frac{W}{E}, \tag{36.20}$$

where the quantity n $(0 < n < 1)$ characterizes the focusing properties of the magnetic field and is known as the fall-off index for the

field near the stable orbit, $H = \text{constant}/r^n$. A. A. Kolomenskiĭ and A. N. Lebedev [36.44] (1956) established that synchrotron radiation may also affect the betatron oscillations, causing radiative damping with a damping factor

$$\Gamma_r = \frac{n}{1-n}\frac{W}{E}, \quad \Gamma_z = \frac{W}{E}. \tag{36.21}$$

Such a strong effect may lead to a situation in which a bunch of electrons being accelerated rapidly decreases in size and becomes a point, with the result that the accelerator will cease to operate thanks to the emergence of extremely strong coherent radiation.

In this respect the quantum fluctuations of the electron trajectory play a salutory role, so to say, since they impart a stochastic spread to the electron beam, thus preventing the electrons from approaching each other without limit. Hence, the electron beam in a cyclic accelerator assumes its natural macroscopic dimensions, which are determined by the balance between the quantum excitation and radiative damping of betatron and phase oscillations.

For small time intervals, when $t \ll \tau = 1/\Gamma_r$, the square of the amplitude of radial oscillations increases according to the law of Brownian motion:

$$a_r^2 = \frac{55}{24\sqrt{3}}\frac{1}{(1-n)^2}\frac{e^2\hbar}{m^2c^2R}\left(\frac{E}{mc^2}\right)^5 t \tag{36.22}$$

(see also formula (36.18) for a uniform magnetic field ($n = 0$)). On the other hand, when $t > \tau$, the quantum growth in the amplitude is suppressed, thanks to radiative damping, and

$$a_r^2 = \frac{55\sqrt{3}}{16}\frac{1}{n(1-n)}\frac{\hbar R}{mc}\left(\frac{E}{mc^2}\right)^2, \tag{36.23}$$

where in the case of a uniform field ($n = 0$) this limit is absent since the damping factor vanishes.

All these ideas formed the basis for calculations involved in the design of electron accelerators with energies of the order of 1 GeV and higher. Most strongly the quantum effects were found to manifest themselves in storage rings, where the accelerated electrons may be confined in a magnetic field for many hours. Allowing for quantum fluctuations in the electron trajectory stimulated the transition to strong focusing, that is, when the fall-off index may assume values that exceed unity considerably.

The effect of quantum fluctuations on the stability of particle motion in a magnetic field, predicted theoretically in [36.43], found substantiation in experiments. For one, in 1959, M. Sands [36.45] discovered that quantum fluctuations in the phase pose serious difficulties in the operation of the Caltech 1.2-GeV synchrotron. Here the breakdown of phase stability in electron motion created difficulties in moving into the energy region above 1 GeV.

Interesting and extremely original studies of the behavior of the electron beam in an accelerator were carried out in 1960 by F. A. Korolev and O. F. Kulikov [36.46], who used the FIAN synchrotron facilities. The idea of the experiment consisted in photographing the beam of particles emitting synchrotron radiation by a high-speed cine camera and subsequently developing the pictures.

The light emitted by the electrons was collected by lenses at a certain point on the optical axis of the system (Figure 36.8) and

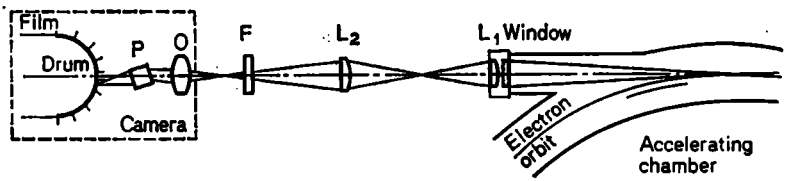

Figure 36.8 Registering an electron bunch

formed into the intermediate image photographed by the high-speed cine camera. The frames were processed on a microphotometer, with photometric measurements carried out in two directions simultaneously: in the radial direction and in the "vertical" direction, that is, in the direction of the magnetic field of the accelerator.

Since during the exposure of a single frame (the camera operated at 5400 frames per minute) an electron in the accelerator performs a large number of oscillations, the picture is averaged out. At the same time the beam is sufficiently monochromatic in energies, whereby the distribution of the relative brightness in the photometric process can be identified with the coordinate distribution of the electrons over the cross section of the beam (bunch). Analyzing the frames in the proper sequence, one can arrive at the law for variation in electron distribution in time by linking this distribution to the amplitude of particle oscillations. It was found, for instance, that it is possible to directly observe the dynamics of the electrons in a cyclic accelerator via the emitted synchrotron radiation.

F. A. Korolev and O. F. Kulikov [36.46] (1960) discovered, in addition to strong radiative damping, quantum fluctuations in the electron trajectory—the excitation of radial oscillations. In the 300- to 400-MeV region of electron energies, the amplitude of radial oscillations $\sigma = (a_{\text{bet}}^2 + a_{\text{syn}}^2)^{1/2}$ was found to differ considerably from value calculated via classical methods (Figure 36.9).

In 1966, F. A. Korolev, O. F. Kulikov, and A. S. Yarov carried out additional studies of quantum effects using the 680-MeV FIAN synchrotron that operated in a special mode [36.47]. In this mode, which was found to be extremely convenient for the experiment, the electron energy remained constant at 285 MeV over a relatively

large period of time (0.5 s). This ensured the constancy of the damping factors and other characteristics related to the electron energy. The optical scheme underwent no serious changes (see Figure 36.8).

The results of the experiments are shown in Figure 36.10. Curve *3* describes the variation of the energy with time: a linear increase up to 285 MeV, with a plateau of a duration of 0.46 s. Curves *1* and *2* represent the experimental and theoretical values of the amplitudes of radial oscillations. The shape of the curves corroborates

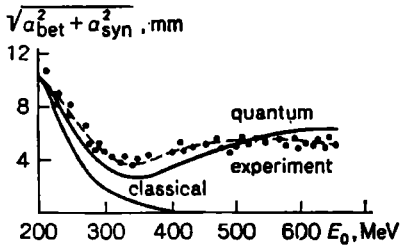

Figure 36.9 Theoretical curves and experimental data on quantum excitation of radial electron oscillations.

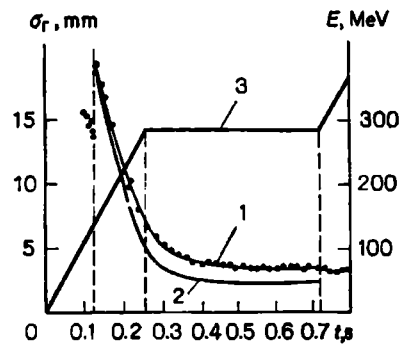

Figure 36.10 The stabilization of radial electron oscillations.

the conclusions made on the basis of quantum theory: on the plateau the damping factors remain constant and the curves become smoother; we also see that the radial oscillations rapidly stabilize on the plateau due to their excitation by quantum fluctuations in the radiation. The amplitude of the steady oscillations agrees fairly well with the value predicted by the theory.

Thus, an effect of stabilization of radial oscillations of the electrons was observed in the experiments (the electron energy remains constant in the process), that is, the quantum fluctuations in the radiation generate the oscillations while radiative damping suppresses them.

Note that soft focusing constitutes the most favorable conditions for observing quantum fluctuations (i.e. $0 < n < 1$). Under strong focusing, $|n| \gg 1$, the amplitude of oscillations in the accelerator drops sharply. In this connection of great interest are the experiments conducted on the 1.5-GeV Tomsk Polytechnical Institute synchrotron constructed under the guidance of A. A. Vorob'ev [36.48]. This accelerator has soft focusing and large acceleration times, which opens up new possibilities for observing quantum effects under conditions of macroscopic motion.

36.4 Spin Effects

Interesting and important results (from the viewpoint of practice) have been obtained in studies on the behavior of electron spin under powerful synchrotron radiation. The development of the quantum theory of this phenomenon has shown (A. A. Sokolov and I. M. Ternov [36.52] (1963)) that synchrotron radiation leads to a directional process of orientation of the electron spin in relation to the external magnetic field, that is, the electron beam becomes polarized.

Indeed, as a result of emission the electron will have the tendency to go over to the state with the smallest possible energy in the magnetic field, which energy is

$$U^{\text{mag}} = -(\boldsymbol{\mu}\cdot\mathbf{H}). \tag{36.24}$$

Therefore, the electron spin tends to point in the direction opposite to that of the magnetic field. For positrons the process is reversed, that is, positron spin will tend to align with the field.

In Section 29 it was shown, via methods of quantum theory, that the time dependence of the average value of the projection of the electron spin on the direction of the external magnetic field is

$$\langle\zeta\rangle = \zeta(t) = -\frac{8\sqrt{3}}{15}\{1-e^{-t/\tau}\}, \tag{36.25}$$

with the polarization time being

$$\tau = \frac{8\sqrt{3}\hbar^2}{15mce^2}\left(\frac{mc^2}{E}\right)^2\left(\frac{H_0}{H}\right)^3, \tag{36.26}$$

where $H_0 = m^2c^3/e_0\hbar = 4.41 \times 10^{13}$ G is the so-called Schwinger magnetic field. Obviously, over a time t greater than the polarization time τ, the electron beam will become 92% polarized irrespective of the initial spin orientation. The polarization time for 1-GeV electrons is about one hour, whereby radiation polarization may be observed in storage rings at great circulation times.

Let us provide a qualitative interpretation for this effect. To this end we will consider the energy balance in the reference frame where the electron is at rest. In this reference frame the spin magnetic moment $\boldsymbol{\mu}$ precesses in accordance with the nonrelativistic equation of motion

$$\frac{d}{dt^0}\boldsymbol{\mu} = -\frac{e_0}{mc}(\boldsymbol{\mu}\times\mathbf{H}^0) = -\omega_0\boldsymbol{\mu}\times\frac{\mathbf{H}^0}{H^0}. \tag{36.27}$$

In this equation, $\mathbf{H}^0$ and t^0 refer to the frame of reference in which the electron is at rest. In view of the fact that the motion of the dipole magnetic moment is oscillatory, the dipole emits radiation,

with the result that the potential magnetic energy of the electron decreases:

$$\frac{d}{dt^0}(-\boldsymbol{\mu}\cdot\mathbf{H}^0) = -\frac{2}{3}\frac{1}{c^3}\ddot{\boldsymbol{\mu}}^2. \tag{36.28}$$

Solving this system of equations, we find that

$$\frac{d}{dt^0}(\boldsymbol{\mu}\cdot\mathbf{H}^0) = -\frac{2}{3}\frac{\omega_0^4}{c^3}\left(\boldsymbol{\mu}\times\frac{\mathbf{H}^0}{H^0}\right)^2. \tag{36.29}$$

Thus, radiation flips the electron spin over, and the energy of this spin-flip is $2\mu_0 H^0 = \hbar\omega_0$, with μ_0 the Bohr magneton. Assuming, as it is commonly done in quasi-classical theory, that $\mu^2 = \mu_0^2(s+1)\,s$, $s = 1/2$, and introducing the dimensionless spin $\boldsymbol{\mu} = -\mu_0\boldsymbol{\zeta}'$, $\zeta'^2 = 3$, from (36.29) we obtain the following:

$$\frac{d}{dt^0}(\boldsymbol{\zeta}'\cdot\mathbf{H}^0) = -\frac{2}{3}\frac{\mu_0\omega_0^4}{c^3}(\zeta^2-\zeta_{\|}'^2) = -\frac{2}{3}\frac{\mu_0\omega_0^4}{c^3}(3-\zeta_{\|}'^2). \tag{36.30}$$

If we now introduce the analog of the quantum projection on the direction of the field, $\zeta_z = \zeta_{\|}'/\sqrt{3}$, then the last equation yields

$$\zeta_z(t) = -\frac{1-e^{-t/T^0}}{1+e^{-t/T_0}}, \quad T^0 = \frac{\sqrt{3}}{2}\frac{\hbar^2}{mce^2}\left(\frac{H_0}{H^0}\right)^3, \tag{36.31}$$

where $H_0 = m^2c^3/e_0\hbar$ is the Schwinger magnetic field. If we now go over to the laboratory system, $T_0 = \gamma^{-1}T$, $H^0 = \gamma H$, $\gamma = E/mc^2$, we find that the polarization time differs only by a factor from the exact quantum result (36.26):

$$T = \frac{\sqrt{3}}{2}\frac{\hbar^2}{mce^2}\left(\frac{H_0}{H}\right)^3\left(\frac{mc^2}{E}\right)^2 \tag{36.32}$$

However, in our extremely simplified consideration of the problem, it takes time $t \gg T$ for the electron beam to become totally polarized: all electron spins become oriented in the direction opposite to that of the magnetic field.

The exact solution of the problem by quantum mechanical methods only gives more precise values for the degree of polarization and the polarization time. Physically the picture is refined by the condition of discrete photon emission by the electrons in the beam. The fluctuations in the radiation manifest themselves in the fact that not only the one-photon ordering process of spin orientation is possible (which is the case with classical theory) but also reverse processes may occur.

The effect of radiation polarization was established in 1963 by A. A. Sokolov and I. M. Ternov [36.52] for the case of electrons and positrons moving in a uniform magnetic field. In the current crop of storage rings the electrons move in a nonuniform focusing magnetic field, which ensures the stable motion of the electrons along the

equilibrium orbit. Accidental deviations from the orbit do not bring the particles out of the state of stable motion. The particles undergo small oscillations, known as betatron oscillations, with frequencies $\omega_r = \sqrt{1-n}\,(c/R)$ and $\omega_z = \sqrt{n}\,(c/R)$ depending on the fall-off index $(0 < n < 1)$ near the equilibrium orbit.

Under such conditions there is the possibility that resonance phenomena may occur. These can destruct the polarization of the beam, a possibility pointed out by V. N. Baier [36.53] (1971). Here the anomalous magnetic moment of the electron, $\mu = \mu_0 (1 + \alpha/2\pi)$, with $\alpha = e^2/\hbar c$, plays an important role, precisely, the spin magnetic moment differs from the Bohr magneton by $\Delta\mu = \mu_0\alpha/2\pi$ (see Section 35).

To analyze the effect that a nonuniform magnetic field has on the electron spin we can either use the Bargmann-Michel-Telegdi equation [36.54], in which spin is interpreted as a classical quantity, or solve this problem by employing the rigorous Dirac theory of electron motion in a magnetic field (A. A. Sokolov, A. V. Borisov, D. V. Gal'tsov, and V. Ch. Zhukovskii [36.55] (1974)). As a result of considering this problem it has been found that in the absence of radiation the projection of the spin on the z axis, the axis of symmetry of the magnetic field, has the form

$$\zeta_z = \cos\Phi + \frac{nB\eta}{2R}\sin\Phi\left\{\frac{\cos(\omega_z+\Omega)t}{\sqrt{n}+\eta} - \frac{\cos(\omega_z-\Omega)t}{\sqrt{n}-\eta}\right\}. \quad (36.33)$$

Here Φ is the initial phase, B and ω_z are the amplitude and frequency of the vertical betatron oscillations of the electron, with $z = B\cos\omega_z t$ and $\omega_z = \sqrt{n}\omega_0$, $\Omega = (\alpha/2\pi)\,(e_0H/mc)$ is the spin precession frequency related to the anomalous magnetic moment, and $\eta = \Omega/\omega_0 = (\alpha/2\pi)\,(E/mc^2)$, with $\omega_0 = c/R$. Note that for the longitudinal polarization, the projection of the spin on the direction of motion, i.e. on the kinetic momentum $\mathbf{P} = \mathbf{p} + (e_0/c)\,\mathbf{A}$, is given by the following formula:

$$\frac{\langle\boldsymbol{\Sigma}\cdot\mathbf{P}\rangle}{|\mathbf{P}|} = \sin\Phi\sin\Omega t + \frac{B}{R}\,n^{3/2}\eta\cos\Phi\,\frac{\sin\omega_z t}{n-\eta^2}. \quad (36.34)$$

Thus, the vacuum magnetic moment of the electron affects not only the longitudinal polarization and makes it unstable but also may serve as a depolarizing factor in relation to the transverse polarization. This happens when the frequency of the betatron oscillations coincides with the spin precession frequency, $\omega_z = \Omega$. This leads to a characteristic resonance ($\eta = \sqrt{n}$), which destroys polarization.

By proper selection of the parameters of the magnetic system one can avoid dangerous resonances and ensure the conditions of

particle motion close to ideal. In these conditions the effect of radiation polarization must manifest itself in exactly the same manner as in a uniform magnetic field. A detailed study of the radiation polarization phenomenon in storage systems designed for super-high electron energies (10-100 GeV) was carried out by Ya. S. Derbenev, A. M. Kondratenko, and A. N. Skrinskii [36.56] (1977). It demonstrated the possibility of forming polarized beams by employing this method.

Note that highly important for current physical experiments is the possibility of using longitudinally polarized electron and positron beams. In a magnetic field the longitudinal polarization is not conserved (see (36.34)), but relatively simple devices enable transforming transverse polarization into longitudinal. For instance, to this end we can switch on a magnetic field parallel to the electron momentum in the straight section of the storage ring. Then the strength of this additional field can be selected such that it rotates the spin through an angle of $\pi/2$, which actually means transferring transverse polarization into longitudinal. Of course, this can be done only at a certain point of the trajectory (usually at the points where the beams meet); at other points the quantum fluctuations destroy the longitudinal polarization and reconstruct the transverse.

Let us dwell on the questions of experimentally observing the effect of radiation polarization. The effect was first observed eight years after its theoretical prediction, in 1971 in Novosibirsk at the Institute of Nuclear Physics of the Siberian Branch of the USSR Academy of Sciences [36.57] polarization was being studied on the 625-MeV VEPP-2 storage ring. In the same year the effect was discovered in France on the 536-MeV ACO storage ring at Orsay (France) [36.58], then in 1975 in the USA at Stanford University with 2.4-GeV electrons [36.60-36.62], and finally in 1980 in the Federal Republic of Germany, on the PETRA machine, with 15.2-GeV electrons [36.63].

The method used in the first experiments in which radiation polarization was observed [36.57, 36.58] was based on Møller scattering of particles by each other in a bunch. This scattering mechanism, related to betatron oscillations (the Touschek effect), may lead to particles being lost if the additional momentum acquired in the scattering process exceeds a certain maximum limit.

Another important aspect is that the scattering cross section depends on the orientation of the particle spin, whereby the number of particles leaving the acceleration mode depends on the polarization of these particles. It was found that the lifetime of a polarized beam is higher than that of nonpolarized. Initially, when the beam is nonpolarized, the number of pairs of particles leaving the beam (particles leave the beam in pairs, as follows from the very meaning of the internal scattering effect) is at a maximum. Then, as radiation polarization proceeds, the spins get ordered and the rate at

which pairs leave the beam drops exponentially. The theoretical formula for this rate has the form

$$Y = a - b(1 - e^{-t/\tau}), \tag{36.35}$$

where the coefficients are such that $a > 0$ and $a > b$, and τ is the polarization time (see (36.26)). At time zero, $t = 0$, we have $Y = Y_{max} = a$, and at $t > \tau$ we have $Y \simeq a - b$. In experiments this exponential law of rate decrease was clearly observed, but then, as time grows, the beam energy changes in such a manner that the electron spin precession frequency, related to the anomalous magnetic moment, acquires a resonance value, that is, becomes equal to the betatron frequency. This is accompanied by a sharp increase in number of pairs of particles leaving the beam, which is proof that the beam is being depolarized. The experiment enables determining the degree of polarization of the beam as well as the polarization time τ. A similar method was used in subsequent experiments on the VEPP-2M machine in Novosibirsk [36.59] and on SPEAR storage ring at Stanford University [36.60].

All these experiments not only qualitatively proved the existence of the radiation polarization effect but demonstrated good agreement with the theoretical prediction concerning the degree of polarization and the polarization time.

The method based on the Touschek effect proved to be invalid for measuring the polarization of colliding beams. Moreover, it is extremely sensitive to the structure of the beam and therefore does not guarantee precision in measuring the polarization.

In another series of experiments a method of measuring the polarization based on the annihilation of particles was employed. For instance, when an electron and a positron collide and annihilate, a pair of μ-mesons is produced: $e^+e^- \to \mu^+\mu^-$. The cross section of this process, as quantum electrodynamics implies, depends on the degree of transverse polarization of the colliding beams:

$$\frac{d\sigma_{\mu\mu}}{d\Omega} = \frac{r_0^2}{16}\left(\frac{mc^2}{E}\right)^2 [1 + z^2 + (1 - z^2) P^2 \cos 2\varphi] \tag{36.36}$$

Here r_0 is the classical electron radius, P is the degree of transverse polarization of the colliding beams, $z = \cos\theta$, with θ the polar angle of the μ^+-meson in relation to the incident positron, and φ is the azimuthal angle formed by the creation plane (the plane in which the momenta of the initial and final particles lie) and the horizontal plane. As the reader can clearly see, the cross section depends on the polarization P of positrons and electrons, which is reflected in its azimuthal asymmetry, or the dependence on φ. In the experiment conducted by J.G. Learned, L.K. Resvanis, and G.M. Spencer [36.61] with 3.7-GeV colliding beams (SPEAR), the results concerning the distribution of events with respect to φ prove

that the beams become polarized. It is characteristic that at 3.1 GeV the dependence of the annihilation cross section on φ disappears: at this energy the spin precession frequency coincides with a harmonic of the orbital frequency, and the polarization of the beams is destroyed because of the depolarizing resonance. The experimental data have shown good agreement with the theoretical results.

Polarization of electron-positron beams caused by synchrotron radiation proved to be highly important for further use of such beams. Here are only two variants of experiments involving such particles.

(1) *The quasi-elastic process* $e^+e^- \to \mu^+\mu^-$. This process may proceed by two channels: the electromagnetic and the weak. For the electromagnetic channel in the particular case of transverse polarization of the electron and positron, formula (36.36) for the cross section is valid.

Selecting the polarization and the angles of observation of the μ-mesons in a special way, the cross section in the electromagnetic channel (one-photon exchange) can be made minimal (as a consequence of parity conservation in electrodynamics). Then it becomes possible to observe relatively amplified effects of the interference of the weak channel with the electromagnetic.

Thus, the polarization of colliding beams is a means for increasing the interference between the weak and electromagnetic amplitudes.

(2) *The annihilation of electron-positron pairs into hadrons*: $e^+e^- \to h + X$. The cross section of annihilation into hadrons with a fixed momentum of one of the hadrons, h, (an inclusive process) also depends on the polarization of the colliding particles. This dependence enables finding independently the structural functions of the hadron (the transverse and the longitudinal) and selecting a particular interaction model (the parton model, the model of charged vector mesons). Thus, experiments involving polarized particles make the annihilation process especially informative.

In 1974, at the Stanford storage ring SPEAR, a group of researchers [36.64, 36.65] carried out important experiments that led to the discovery of new charmed ψ-particles. These particles belong to the group of strongly interacting unstable bosons (hadrons) with masses lying between 3 and 4 GeV. They are formed as a result of the annihilation of colliding electrons and positrons.

The remarkable fact that the new particles are so stable, which is by no means characteristic of heavy particles, drew great attention to the discovery. The decay of ψ-particles into common hadrons proved to be strongly suppressed, which can be explained by some sort of new quantum numbers characterizing the particles, numbers with their own selection rules. In this connection it was suggested to expand the quark model by introducing a fourth quark with fractional charge 2/3 and carrying a new quantum number, which became known as "charm".

Closely related to the question of experimentally verifying these hypotheses, and especially the quark model, are the experiments on jets of particles. There exists an assumption that hadrons in the final state appear from a quark-antiquark pair. This pair is formed from a virtual photon, with the quark and antiquark (Figure 36.11)

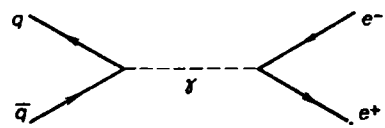

Figure 36.11

behaving like free Dirac particles, and only then are hadrons created. An important aspect of all this is the fact that in the center-of-mass system hadron creation occurs with low transverse momenta in relation to the direction of the quark momenta. Because of this the hadron distribution has the elongated shape of a jet (Figure 36.12).

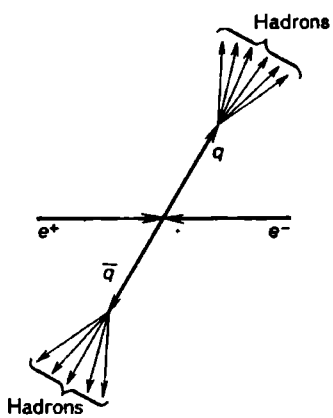

Figure 36.12

In this connection it is also important to note that the polarization of the electron-positron beams must generate an azimuthal asymmetry of the outgoing hadrons with respect to the direction of the initial beams. This was discovered in experiments. Thus, polarization of electron-positron beams has proved highly important for the experimental verification of the modern theory of strong interactions, quantum chromodynamics.

In the summer of 1980 the effect of radiation polarization was observed on the PETRA storage ring (FRG) [36.63]. The measuring method was based on the effect of Compton scattering of laser photons by electrons, with the scattering cross section depending on the orientation of the electron and positron spins. The experiments revealed the polarization of particles at 15.2 GeV.

There is also another interesting possibility for observing the polarization of electrons and positrons, namely, by the spectrum of synchrotron radiation these particles emit [36.66]. This constitutes a direct method, since it requires no other particles or fields acting on the electron (or positron). Indeed, as is known (see Section 29), the radiated power of synchrotron radiation depends on the orien-

tation of the particle spin. Comparing the radiated power for a polarized and a nonpolarized electron beam in the part of the spectrum that is most suitable for observation, the researcher is able to obtain data on the degree of polarization of the particles.

In conclusion we wish to note that the effect of radiation polarization of electrons and positrons in storage rings is at present the only way of obtaining polarized beams of high-energy particles.

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Conclusion

We have thus expounded the main principles of classical and quantum field theories using one of the known types of interaction, the electromagnetic, as an example. We tried to stress the importance of two basic moments in constructing a field theory, the Lagrange-Hamilton method and the symmetry principles. Both have been developed still further in the last years. After more than twenty years of attempts to bury the Hamilton method by means of the theory of Regge poles, the method of dispersion relations, and the axiomatic quantum field theory, it has recreated itself in the new theory of strong interactions, quantum chromodynamics. The symmetry principle was placed at the base of both quantum chromodynamics and unified theories of elementary particle interactions. More than that, it is not an overstatement to say that the majority of the large accomplishments of modern physics are related to symmetries. Suffice it to note the theoretical description of second-order phase transitions, examples of which are the transition of metals into the superconducting state and the transition of magnetic substances into the ferromagnetic state. Most important for field theory are local symmetries, which determine not only the kinematics of fundamental processes but also their dynamics. This principle lies at the base of the theory of gravity created by A. Einstein, the general theory of relativity, whose equations are invariant under local transformations of coordinates. At the same time, the theory of electromagnetism, or electrodynamics, is also based on a local symmetry, precisely, on the U (1) gauge group of local phase transformations.

In the last two decades the idea of local gauge symmetry has generated a new class of the so-called unified theories describing the interaction of elementary particles. The most successful of these was found to be the Weinberg-Salam-Glashow theory, which establishes a relation between the electromagnetic and weak interactions and is based on the broken local symmetry $SU(2) \times U(1)$. The fact that this group of inner symmetry is local requires the existence of four vector fields: two neutral and two charged. The corresponding neu-

tral particles, the photon and the Z^0-boson, and the charged particles, the W^+- and the W^--boson, are the carriers of interaction between leptons. Three of these particles, Z^0, W^+, and W^-, are massive and are responsible for the weak interactions. These particles were discovered in 1983 in experiments conducted at CERN involving colliding proton-antiproton beams, and the masses of the particles were found to be in excellent agreement with the values predicted by the theory. Another unified theory describes the unification of three fundamental interactions out of the four, namely, the electromagnetic, the weak, and the strong; and the scheme became known as the grand unification theory. The idea of grand unification is based on the assumptions that the effective coupling constants for the different gauge interactions, the electromagnetic, the weak, and the strong, depend on the distance between the particles. For instance, the electromagnetic constant grows as the distance diminishes, while the strong and weak coupling constants drop. This phenomenon is related to the different manifestations of vacuum polarization in the Abelian electromagnetic theory and in the non-Abelian theories of weak and strong interactions. As a result, at relatively small distances, or at high energies of the particles (which is the same thing), these constants may become equal to each other, and we will arrive at unified fundamental interaction. Such models of grand unification predict the existence of new fields and new particles, but the most important prediction is the one concerning the decay of the proton, a particle that was always considered stable. Although the lifetime predicted by the theory is colossal, greater than 10^{31} years, the decay is being intensively sought for in many laboratories over the world. If such an event were observed, we would be sure of the validity of the unified quantum field theory of electroweak and strong interactions.

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In their classical work on the theory of synchrotron radiation, Prof. Sokolov and Prof. Ternov explained fundamental properties of this phenomenon, including the quantum fluctuations in synchrotron radiation, the excitation of betatron and synchrotron oscillations in cyclic accelerators, and the radiation polarization of electrons caused by synchrotron radiation (the Sokolov-Ternov effect, 1963). At present Prof. Ternov is investigating the electromagnetic and weak interactions of elementary particles moving in strong external fields.

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Quantum Electrodynamics

The book contains a systematic presentation of the basics of the classical and quantum theories of fields. It thoroughly treats the question of symmetry groups, with the property of gauge symmetry used to describe the electromagnetic field and Yang-Mills fields. Quantization of the electromagnetic field is presented within the framework of Coulomb and Lorentz gauges. Invariant perturbation theory is discussed and Feynman's rules are formulated. Examples are considered that illustrate how to apply the tools of quantum electrodynamics in calculations of electrodynamic effects, namely, the Compton effect bremsstrahlung, and quantum effects in synchrotron radiation.

The authors wrote this text on the basis of lectures delivered at the Physics Faculty of Moscow State University. It is aimed at students majoring in physics, but can also be used as a textbook for institutions of higher learning where a course in theoretical physics is read.